Algorithmic Computability of the Signal Bandwidth

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***Abstract*—The bandwidth of a bandlimited signal is an important number that is relevant in many applications and concepts. For example, according to the Shannon sampling theorem, the bandwidth determines the minimum sampling rate that is required for a perfect reconstruction. In this paper we consider bandlimited signals with finite energy and bandlimited signals that are absolutely integrable and analyze whether the bandwidth of these signals can be determined algorithmically. We employ the concept of Turing computability, a theoretical model that describes the fundamental limits of what can be solved algorithmically on a digital hardware, and ask if, for a given computable bandlimited signal, it is possible to compute its bandwidth on a Turing machine. We show that this is not possible in general, because there exist computable bandlimited signals for which the bandwidth is a non-computable real number. Even the weaker question if the bandwidth of a given signal is smaller than a predefined value cannot be always answered algorithmically. Further, we prove that in the case where the bandwidth in not computable, it is even impossible to algorithmically determine a sequence of upper bounds that converges to the actual bandwidth of the signal. As a positive result, we show that the set of signals whose bandwidth is larger than some given value is semi-decidable.**

***Index Terms*—Bandlimited signal, bandwidth, minimum sampling rate, algorithmic solvability, Turing computability**

I. Introduction

BANDLIMITED signals are important in many applications [2]–[5]. In information theory and signal processing they provide a mathematical framework, in which it is possible to convert analog continuous-time signals into discrete-time signals without a loss of information [6]–[8]. Further, bandlimited signals play an important role in the theory of system approximation processes [9], [10] and in wireless communication systems, where the spectrum of the transmit signal has to be confined to certain bands, in order not to disturb other users [11], [12].

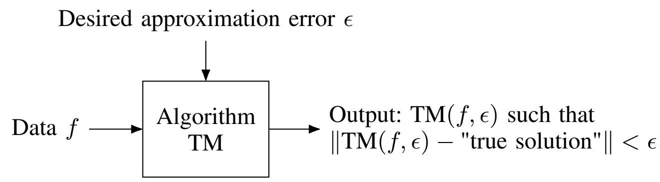

Fig. 1. The algorithm TM gets two inputs: the data f and the desired approximation error ϵ.

The actual bandwidth $B(f)$ of a bandlimited signal f, i.e., the smallest number σ such that f is bandlimited with bandwidth σ, is a relevant quantity in information theory and signal processing [13]–[17], because it is directly linked to the smallest sampling rate r_{min} that is needed in order that the sequence of samples $\{f(k/r_{\text{min}})\}_{k\in\mathbb{Z}}$ uniquely determines f. According to Shannon's famous sampling theorem [6], [18], a bandlimited signal f with finite energy can be reconstructed from its samples $\{f(k/r)\}_{k\in\mathbb{Z}}$, by means of the Shannon sampling series, if $r \geq r_{\text{min}} = B(f)/\pi$. The sequence of samples $\{f(k\pi/B(f))\}_{k\in\mathbb{Z}}$ can therefore be seen as a minimum representation of the signal f. Finding minimum representations of bandlimited signals is essential for the digital transformation and information theory, where it is crucial to know whether a chosen sampling rate is sufficient to completely describe a signal [19]. In information theory, the minimum, i.e., Nyquist sampling rate, is also the basis for the interpolation of discrete-time independent and identically distributed random variables [7], [8]. The Nyquist rate is the largest sampling rate such that the samples of the signal can be chosen independently. For any larger sampling rate, i.e., with oversampling, there are dependencies between the samples.

Digital computers are widely used for various tasks. In this paper we study whether the bandwidth of a computable bandlimited signal can always be determined algorithmically on a digital computer. To this end we employ the concept of Turing computability, a theoretical model that describes the fundamental limits of computation on a digital computer.

It is known that there exist problems that cannot be solved on a digital computer, e.g., the computation of the Fourier transform for certain signals [20], [21] or the spectral factorization [22]. For these signals, a digital computer cannot produce, for every desired error, a result within the error margin, i.e., the approximation error cannot be controlled.

Today, most simulations are done without explicitly treating the approximation error. This, for example, can be observed in the absence of error bars in plots. In those cases, the computer computes an approximation of the solution, however, without any quality guarantees. Such quality guarantees, in the

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form of an algorithmic control of the approximation error, are exactly provided by the concept of computability. In addition to the data f, the desired approximation error ϵ, which could be the maximum tolerable error, is given as an input to the algorithm. The computer continues its computations until it has produced a solution that satisfies this error specification. This algorithmic control of the error is illustrated in Fig. 1.

In this paper we will study the algorithmic computability of the actual bandwidth $B(f)$ of computable bandlimited signals. In particular, we will analyze and answer the questions:

- Question 1: Is $B(f)$ always computable?
- Question 2: Can we algorithmically compute asymptotically sharp lower bounds for $B(f)$?
- Question 3: Can we algorithmically compute asymptotically sharp upper bounds for $B(f)$?

In terms of sampling rate, those questions read: Is the optimal, i.e. minimum required sampling rate always computable? Can we compute lower bounds for the minimum required sampling rate? Can we compute upper bounds for the minimum required sampling rate?

In Section II we introduce the basic concepts of computability theory and the facts that are relevant for us, and, in Section III, the essentials about bandlimited signals are presented. In Section IV we define computable bandlimited signals. Our first main result is given in Section V, where we prove for the signal spaces $\mathcal{B}_\pi^1$ and $\mathcal{B}_\pi^2$ that Question 1 has to be answered in the negative. In Section VI we analyze whether it is possible to algorithmically detect the signals f for which $B(f)$ is not computable. Again, the answer is no. Then in Section VII we study the semi-decidability of certain sets. We show that it is always possible to algorithmically detect if the actual bandwidth of a signal $B(f)$ is larger than a predefined value. The opposite problem, however, to detect if the actual bandwidth of a signal $B(f)$ is smaller than a predefined value, cannot always be solved algorithmically. In VIII we analyze Questions 2 and 3. It will turn out that Question 2 can be answered positively, while Question 3 has to be answered in the negative. Finally, we conclude the paper in Section IX with a discussion and further open problems in Section X.

II. General Algorithms and Computability

The theory of computability is a well-established field in computer sciences [23]–[27]. However, since computability is less well known in the information theory community, we describe some of the key concepts in this section. For a more detailed treatment of the topic, see for example [25]–[28].

In order to study the question of computability, we employ the concept of Turing computability. A Turing machine is an abstract device that manipulates symbols on a strip of tape according to certain rules [23]–[25], [27]. Although the concept is very simple, a Turing machine is capable of simulating any given algorithm. Turing machines have no limitations in terms of memory or computing time, and hence provide a theoretical model that describes the fundamental limits of any practically realizable digital computer. Moreover, Turing machines are equivalent to other concepts of computability, such as those defined by general recursive functions, Minsky register machines, and λ-calculus [27], [29]. It will become clear that Turing computability is exactly the concept that characterizes what can be theoretically achieved by digital hardware, e.g., central processing units (CPUs), digital signal processors (DSPs), or field programmable gate arrays (FPGAs), if practical limitations, such as energy constraints, computing errors, and hardware restrictions, are disregarded.

It is important to distinguish Turing computability from complexity theory, another topic in computer science. Complexity theory deals with the question of how efficiently a problem can be solved, and analyzes how the computation time of a given algorithm scales with the size of the input data. Thus, the goal of complexity theory is different from the goal in Turing computability, where the fundamental limits of computability are explored, without consideration of complexity issues. Further, complexity theory operates in a discrete and finite setting. However, in the modeling of many real world problems continuous signals are used, e.g., bandlimited signals that have an infinite duration. Thus, in order to be able to apply complexity theory on such “continuous problems”, it is necessary that the continuous signals can be approximated by discrete and finite signals in a controlled way, where the approximation error is Turing computable.

A *recursive function* is a function, mapping natural numbers into natural numbers, that is built of simple computable functions and recursions. We will not go into details here, for further information on recursive functions see for example [30]. For us it is important that recursive functions are computable by a Turing machine. A set $\mathcal{A} \subseteq \mathbb{N}$ is called *recursively enumerable* if $\mathcal{A} = \emptyset$ or $\mathcal{A}$ is the range of a recursive function. A set $\mathcal{A} \subseteq \mathbb{N}$ is called *recursive* if both $\mathcal{A}$ and $\mathbb{N} \setminus \mathcal{A}$ are recursively enumerable.

Definition 1. We say that a set $\mathcal{A} \subsetneq \mathbb{N}$ is a *recursively enumerable non-recursive* set, if $\mathcal{A}$ is recursively enumerable but not recursive, i.e., if $\mathcal{A}$ is recursively enumerable but $\mathbb{N} \setminus \mathcal{A}$ is not recursively enumerable.

Such recursively enumerable non-recursive sets exist [30, 4.4 Proposition, p. 19] and will be of great importance for the results in this paper. For every recursively enumerable non-recursive set $\mathcal{A} \subsetneq \mathbb{N}$, there exists a recursive enumeration of $\mathcal{A}$, i.e., a recursive function $\phi_{\mathcal{A}} \colon \mathbb{N} \to \mathcal{A}$ that is surjective and injective.

Alan Turing introduced the concept of a computable real number in [23], [24]. Our definition of a computable real number is based on computable sequences of rational numbers.

Definition 2. A sequence of rational numbers $\{r_n\}_{n\in\mathbb{N}}$ is called *computable sequence* if there exist recursive functions a, b, s from $\mathbb{N}$ to $\mathbb{N}$ such that $b(n) \neq 0$ for all $n \in \mathbb{N}$ and

$$r_n = (-1)^{s(n)} \frac{a(n)}{b(n)}, \quad n \in \mathbb{N}.$$

Definition 3. A real number x is said to be *computable* if there exist a computable sequence of rational numbers $\{r_n\}_{n\in\mathbb{N}}$ and a recursive function $\xi \colon \mathbb{N} \to \mathbb{N}$ such that for all $N \in \mathbb{N}$ we have $|x - r_n| \leq 2^{-N}$ for all $n \geq \xi(N)$. By $\mathbb{R}_c$ we denote the set of computable real numbers and by $\mathbb{C}_c = \mathbb{R}_c + i\mathbb{R}_c$ the set of computable complex numbers.

Note that the recursive, i.e., computable function ξ allows us to control the approximation error algorithmically. This form of convergence, where we have a computable control of the approximation error is called *effective convergence*. $\mathbb{R}_c$ is a field, i.e., finite sums, differences, products, and quotients of computable numbers are computable. Note that commonly used constants like e and π are computable. A non-computable real number was for example constructed in [31].

In this paper, the series

$$\sum_{k=1}^{\infty} \frac{1}{2^{\phi_{\mathcal{A}}(k)}},$$

where $A \subsetneq \mathbb{N}$ is a recursively enumerable non-recursive set, will play an important role. We discuss the relevant properties next. First, note that

$$a_l = \sum_{k=1}^{l} \frac{1}{2^{\phi_{\mathcal{A}}(k)}} \leq \sum_{k=1}^{l} \frac{1}{2^k} \tag{1}$$

for all $l \in \mathbb{N}$, because the numbers $\{\phi_{\mathcal{A}}(k) \colon k = 1, \ldots, l\}$ in general differ from the numbers $\{1, \ldots, l\}$, which maximize the sum on the right-hand side of (1). Therefore, we see that

$$\sum_{k=1}^{\infty} \frac{1}{2^{\phi_{\mathcal{A}}(k)}} \leq \sum_{k=1}^{\infty} \frac{1}{2^k} = 1. \tag{2}$$

Hence, $\{a_l\}_{l \in M}$ is a monotonically increasing and bounded sequence of real numbers. According to the monotone convergence theorem, this sequence has a well-defined limit

$$\omega_* = \lim_{l \to \infty} a_l = \sum_{k=1}^{\infty} \frac{1}{2^{\phi_{\mathcal{A}}(k)}},$$

where $\omega_* \in \mathbb{R}$. However, it can be shown that $\omega_* \notin \mathbb{R}_c$ [26, Corollary 2b, p. 20]. This fact will be important for us.

Lemma 1. *We have $\omega_* \notin \mathbb{R}_c$, i.e., ω_* is a non-computable real number.*

There are several—not equivalent—definitions of computable functions, most notably, computable continuous functions, Turing computable functions, Markov computable functions, and Banach–Mazur computable functions [28]. An example of a function, which is not Turing computable was given in [32]. A function that is computable with respect to any of the above definitions, has the property that it maps computable numbers into computable numbers. This property is therefore a necessary condition for computability. Usual functions like $\sin$, sinc, $\log$, and $\exp$ are computable with respect to all above definitions, and finite sums of computable functions are computable [26].

We will use a definition of a computable function that is based on the idea of effective approximation. As atoms in the approximation we use very basic functions that are computable. A more complicated function f is called computable if it can be effectively approximated by finite linear combinations of the atoms, or, in other words, if there exists an algorithm that, for every approximation error $\epsilon > 0$, can compute, in a finite number of steps, an approximation of f by using only finite linear combinations of the atoms, such that the approximation error is guaranteed to be less than ϵ.

III. Bandlimited Signals

Next, we introduce the necessary definitions. In Appendix A, a list of all important symbols and sets is included. For $\Omega \subseteq \mathbb{R}$, let $L^p(\Omega)$, $1 \leq p < \infty$, be the space of all measurable, pth-power Lebesgue integrable functions on Ω, with the usual norm $\|\cdot\|_p$, and $L^\infty(\Omega)$ the space of all functions for which the essential supremum norm $\|\cdot\|_\infty$ is finite [33].

A function f is said to be *entire* if it is defined and holomorphic on all of $\mathbb{C}$. We employ the usual definition of a bandlimited function that is based on entire functions. Recently, generalizations have been proposed, for example, in [34], where the notion of variable bandwidth was studied.

Definition 4. An entire function f is called *bandlimited* with bandwidth $0 \leq \sigma < \infty$ if for all $\epsilon > 0$ there exists a constant $C(\epsilon)$ with

$$|f(z)| \leq C(\epsilon)\, \mathrm{e}^{(\sigma+\epsilon)|z|} \tag{3}$$

for all $z \in \mathbb{C}$ [18], [35]. By $\mathcal{B}_\sigma$ we denote the set of all entire functions that are bandlimited with bandwidth σ.

Note that, according to this definition, $f \in \mathcal{B}_{\sigma_1}$ implies that $f \in \mathcal{B}_{\sigma_2}$ for all $\sigma_2 \geq \sigma_1$. Thus, a signal that is bandlimited with bandwidth σ_1 is also bandlimited with any bandwidth σ_2 larger than σ_1. For a given bandlimited signal f we denote by

$$B(f) = \min\{\sigma \geq 0 \colon f \in \mathcal{B}_\sigma\}$$

the minimum bandwidth of the signal f, which we will call the *actual bandwidth* of the signal f in the following. In Appendix B, we will prove that this minimum always exist.

Definition 5. The *Bernstein space* $\mathcal{B}_\sigma^p$, $\sigma \geq 0$, $1 \leq p \leq \infty$, consists of all functions in $\mathcal{B}_\sigma$, whose restriction to the real line is in $L^p(\mathbb{R})$ [18, p. 49]. The norm for $\mathcal{B}_\sigma^p$ is given by the L^p-norm on the real line, i.e., by

$$\|f\|_{\mathcal{B}_\sigma^p} = \left(\int_{-\infty}^{\infty} |f(t)|^p \, \mathrm{d}t \right)^{\frac{1}{p}}.$$

$\mathcal{B}_{\sigma,0}^\infty$ denotes the space of all functions in $\mathcal{B}_\sigma^\infty$ that vanish on the real axis at infinity.

Remark 1. We have $\mathcal{B}_\sigma^r \subseteq \mathcal{B}_\sigma^s \subseteq \mathcal{B}_{\sigma,0}^\infty$ for all $1 \leq r \leq s < \infty$.

Let $\hat{f}$ denote the Fourier transform of a signal f. For $f \in L^1(\mathbb{R})$, we have

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)\, \mathrm{e}^{-i\omega t} \, \mathrm{d}t.$$

For $f \in L^2(\mathbb{R})$ the Fourier transform is defined in L^1 and extended canonically to L^2. For $f \in L^p(\mathbb{R})$, $p > 2$, more refined definitions that are based on distribution theory have to be employed [36].

$\mathcal{B}_\sigma^2$ is the frequently used space of bandlimited signals with finite energy. According to the Paley–Wiener theorem [18, Theorem 7.2, p. 68], the support of the Fourier transform $\hat{f}$ of a signal $f \in \mathcal{B}_\sigma^2$ is contained in $[-\sigma, \sigma]$, and we have

$$f(t) = \frac{1}{2\pi} \int_{-\sigma}^{\sigma} \hat{f}(\omega)\, \mathrm{e}^{i\omega t} \, \mathrm{d}\omega.$$

Hence, for the space $\mathcal{B}_\sigma^2$ we have a further, different characterization of the actual bandwidth. For $f \in \mathcal{B}_\sigma^2$, $B(f)$ is the smallest number $\eta > 0$ such that

$$f(t) = \frac{1}{2\pi} \int_{-\eta}^{\eta} \hat{f}(\omega)\, \mathrm{e}^{i\omega t}\ \mathrm{d}\omega \tag{4}$$

for all $t \in \mathbb{R}$. According to Plancherel's identity, this is also the smallest $\eta > 0$ such that

$$\int_{-\infty}^{\infty} |f(t)|^2\ \mathrm{d}t = \frac{1}{2\pi} \int_{-\eta}^{\eta} |\hat{f}(\omega)|^2\ \mathrm{d}\omega. \tag{5}$$

For spaces other than $\mathcal{B}_\sigma^2$, the characterizations of the actual bandwidth via (4) and (5) get mote involved, since, in general, the Fourier transform has to be defined via distributions. However, for the space $\mathcal{B}_\sigma^1$ we still have the characterizations (4) and (5).

The actual bandwidth $B(f)$ of a bandlimited signal f is a distinguished quantity, because it determines the minimum sampling rate that is required so that the samples uniquely determine f. This follows, as we will see, directly from the Plancherel–Pólya inequality [35, p. 152]. Without loss of generality, we consider $\sigma = \pi$, i.e., signals that are bandlimited to π, in the following.

Theorem (Plancherel–Pólya)**.** *Let $1 < p < \infty$. Then there exist two constants $C_L > 0$ and $C_R > 0$, depending only on p, such that for all $f \in \mathcal{B}_\pi^p$ we have*

$$C_L \left(\sum_{k=-\infty}^{\infty} |f(k)|^p \right)^{\frac{1}{p}} \leq \|f\|_p \leq C_R \left(\sum_{k=-\infty}^{\infty} |f(k)|^p \right)^{\frac{1}{p}}. \tag{6}$$

According to the Plancherel–Pólya inequality, for signals $f \in \mathcal{B}_\pi^p$, $1 < p < \infty$, $\mathbb{Z}$ is a set of uniqueness, i.e., f is uniquely determined by its samples $\{f(k)\}_{k\in\mathbb{Z}}$. Further, the right inequality of (6) implies that $\mathbb{Z}$ is a *set of stable sampling* [37]–[39]. $\mathbb{Z}$ is also a *set of interpolation* for $\mathcal{B}_\pi^p$, $1 < p < \infty$, because for every sequence $\alpha = \{\alpha_k\}_{k\in\mathbb{Z}} \in \ell^p$ we can find a signal $f \in \mathcal{B}_\pi^p$ such that $f(k) = \alpha_k$ for all $k \in \mathbb{Z}$. If the sampling rate is reduced below the critical Nyquist rate, i.e., if sampling points $a\mathbb{Z}$ with $a > 1$ are considered, then $a\mathbb{Z}$ is no longer a set of uniqueness for signals in $\mathcal{B}_\pi^p$, $1 < p < \infty$, in general. We give the proof of this fact in Appendix C. Hence, for signals f with arbitrary bandwidth $\sigma > 0$, it follows that $B(f)$ is the smallest number such that the sequence of samples $\{f(k\pi/B(f))\}_{k\in\mathbb{Z}}$ completely determines $f \in \mathcal{B}_\sigma^p$, $1 < p < \infty$. That is, $B(f)$ determines the optimum sampling rate, and consequently the minimum storage rate that is required to reconstruct f.

The Plancherel–Pólya inequality couples the continuous L^p-norm of a bandlimited signal with the discrete ℓ^p-norm of its samples and therefore is an ideal tool to study the conversion from the continuous-time domain into the discrete-time domain and vice versa. Both are essential steps in the digital transformation, where analog signals are processed in the digital domain. The coupling of the norms can be used to prove the convergence of the Shannon sampling series. For $f \in \mathcal{B}_\pi^p$, $1 < p < \infty$, the Shannon sampling series

$$\sum_{k=-\infty}^{\infty} f(k) \frac{\sin(\pi(t-k))}{\pi(t-k)}$$

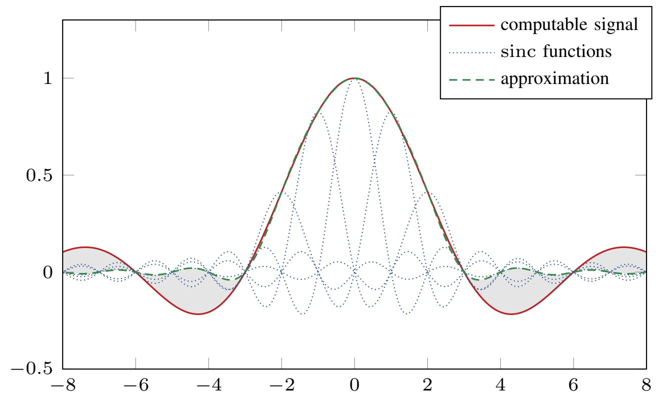

Fig. 2. Approximation of a computable signal (solid red) by an elementary computable function (dashed green). The elementary computable function (dashed green) is the finite sum of five sinc functions (dotted blue). The approximation error is indicated by the gray area.

converges to f in the $\mathcal{B}_\pi^p$-norm, as can be easily seen: For $N_1 > N_2$ we have

$$\begin{aligned}\Bigg(\int_{-\infty}^{\infty} \Bigg| \sum_{k=-N_1}^{N_1} f(k) \frac{\sin(\pi(t-k))}{\pi(t-k)} \\ - \sum_{k=-N_2}^{N_2} f(k) \frac{\sin(\pi(t-k))}{\pi(t-k)} \Bigg|^p \mathrm{d}t \Bigg)^{\frac{1}{p}} \\ \leq C_R \Bigg(\sum_{N_2 < |k| \leq N_1} |f(k)|^p \Bigg)^{\frac{1}{p}},\end{aligned}$$

where we used the Plancherel–Pólya inequality. Since $\{f(k)\}_{k\in\mathbb{Z}} \in \ell^p$, it follows that

$$\left\{ \sum_{k=-N}^{N} f(k) \frac{\sin(\pi(\,\cdot\,-k))}{\pi(\,\cdot\,-k)} \right\}_{N\in\mathbb{N}}$$

is a Cauchy sequence in $\mathcal{B}_\pi^p$. This implies that the Shannon sampling sequence converges to f in the $\mathcal{B}_\pi^p$-norm.

IV. Computable Bandlimited Signals

Before we come to our main result, we need to introduce the concept of a computable function in a Banach space. The definition will employ the idea of effective approximability that was introduced at the end of Section II. In order to be more specific, we discuss computability in the context of bandlimited signals for the spaces $\mathcal{B}_\pi^p$, $p \in [1,\infty) \cap \mathbb{R}_c$ and $\mathcal{B}_{\pi,0}^\infty$.

Definition 6. We call a function f *elementary computable* if there exists a natural number L and a sequence of computable numbers $\{\alpha_k\}_{k=-L}^{L}$ such that

$$f(t) = \sum_{k=-L}^{L} \alpha_k \frac{\sin(\pi(t-k))}{\pi(t-k)}. \tag{7}$$

The building blocks of an elementary computable function are sinc functions. Hence, elementary computable functions are exactly those functions that can be represented by a finite Shannon sampling series with computable coefficients

$\{\alpha_k\}_{k=-L}^{L}$. Note that every elementary computable function f is a finite sum of computable continuous functions and hence a computable continuous function. As a consequence, for every $t \in \mathbb{R}_c$ the number $f(t)$ is computable. Further, the sum of finitely many elementary computable functions is elementary computable, as well as the product of an elementary computable function with a computable number $\lambda \in \mathbb{C}_c$. Hence, the set of elementary computable functions is closed with respect to the operations addition and multiplication with a scalar. Further, for every elementary computable function f, the norm $\|f\|_{\mathcal{B}_\pi^p}$, $p \in (1,\infty)\cap\mathbb{R}_c$, is a computable real number.

Definition 7. A signal in $f \in \mathcal{B}_\pi^p$, $p \in [1,\infty) \cap \mathbb{R}_c$, is called *computable in* $\mathcal{B}_\pi^p$ if there exists a computable sequence of elementary computable functions $\{f_n\}_{n\in\mathbb{N}}$ and a recursive function $\xi\colon \mathbb{N} \to \mathbb{N}$ such that for all $N \in \mathbb{N}$ we have

$$\|f - f_n\|_{\mathcal{B}_\pi^p} \leq \frac{1}{2^N} \tag{8}$$

for all $n \geq \xi(N)$. We use the same definition for signals in $\mathcal{B}_{\pi,0}^\infty$. By $\mathcal{CB}_\pi^p$, $p \in [1,\infty)\cap\mathbb{R}_c$, we denote the set of all signals in $\mathcal{B}_\pi^p$ that are computable in $\mathcal{B}_\pi^p$, and by $\mathcal{CB}_{\pi,0}^\infty$ the set of all signals in $\mathcal{B}_{\pi,0}^\infty$ that are computable in $\mathcal{B}_{\pi,0}^\infty$.

According to this definition we can approximate any signal $f \in \mathcal{CB}_\pi^p$, $p \in [1,\infty) \cap \mathbb{R}_c$, by an elementary computable signal, where we have an "effective", i.e. computable control of the approximation error. For every prescribed approximation error $\epsilon > 0$, $\epsilon \in \mathbb{R}_c$, we can compute an index $N = \lceil -\log_2(\epsilon)\rceil$ such that the approximation error $\|f - f_n\|_{\mathcal{B}_\pi^p}$ is less than or equal to ϵ for all $n \geq \xi(N)$. Hence, the type of convergence that we have in (8) is called effective convergence. In Fig. 2 the approximation of a computable signal by an elementary computable signal is illustrated. Note that $\mathcal{CB}_\pi^p$, $p \in [1,\infty) \cap \mathbb{R}_c$, and $\mathcal{CB}_{\pi,0}^\infty$ have a linear structure.

Remark 2. Due to the inequality

$$\left|\|f\|_{\mathcal{B}_\pi^p} - \|f_n\|_{\mathcal{B}_\pi^p}\right| \leq \|f - f_n\|_{\mathcal{B}_\pi^p},$$

it follows immediately that the norm $\|f\|_{\mathcal{B}_\pi^p}$ is a computable real number for all $f \in \mathcal{CB}_\pi^p$, $p \in [1,\infty) \cap \mathbb{R}_c$. See also [26, pp. 40].

Since, for $f \in \mathcal{CB}_\pi^p$, $p \in [1,\infty) \cap \mathbb{R}_c$, we have

$$|f(t)| \leq \left(\sum_{k=-\infty}^{\infty} |f(t+k)|^p\right)^{\frac{1}{p}} \leq (1+\pi)\|f\|_{\mathcal{B}_\pi^p}$$

for all $t \in \mathbb{R}$, where we used Nikol'skiĭ's inequality [18, p. 49] in the last inequality, we see that

$$\|f\|_\infty \leq (1+\pi)\|f\|_{\mathcal{B}_\pi^p} \tag{9}$$

for all $f \in \mathcal{CB}_\pi^p$, $p \in [1,\infty) \cap \mathbb{R}_c$. Hence, the pointwise approximation error is smaller than or equal to the approximation error measured in the $\mathcal{B}_\pi^p$-norm. This means that we can also control the pointwise approximation error.

We can use inequality (9) to obtain the following fact about relation of the sets $\mathcal{CB}_\pi^p$, $p \in [1,\infty) \cap \mathbb{R}_c$, and $\mathcal{CB}_{\pi,0}^\infty$.

Fact 1. *If* $f \in \mathcal{CB}_\pi^p$, $p \in [1,\infty) \cap \mathbb{R}_c$, *then we also have* $f \in \mathcal{CB}_{\pi,0}^\infty$.

Proof. Let $f \in \mathcal{CB}_\pi^p$, $p \in [1,\infty] \cap \mathbb{R}_c$. Then there exists a computable sequence of elementary computable functions $\{f_n\}_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty}\|f - f_n\|_{\mathcal{B}_\pi^p} = 0$, where the convergence is effective. Due to (9), $\{f_n\}_{n\in\mathbb{N}}$ also converges effectively to f in the $\mathcal{B}_{\pi,0}^\infty$-norm, which implies that $f \in \mathcal{CB}_{\pi,0}^\infty$. □

In order to have a meaningful definition of a computable signal in $\mathcal{B}_\pi^p$, $p \in (1,\infty) \cap \mathbb{R}_c$, it is necessary that each $f \in \mathcal{B}_\pi^p$ can be approximated in a classical sense by a linear combination of shifted sinc-functions. This is assured by the next fact. Note that for $p = 2$ the fact is just the statement of the Shannon sampling theorem. For the general theory, i.e., $p \neq 2$, see, for example, [35, Theorem 3, p. 152].

Fact 2. *Let* $f \in \mathcal{B}_\pi^p$, $p \in (1,\infty)$. *For every* $\epsilon > 0$ *there exists an* $L \in \mathbb{N}$ *and numbers* $\{\alpha_k\}_{k=-L}^{L}$ *such that*

$$\left\| f - \sum_{k=-L}^{L} \alpha_k \frac{\sin(\pi(\,\cdot\, - k))}{\pi(\,\cdot\, - k)} \right\|_{\mathcal{B}_\pi^p} < \epsilon.$$

Remark 3. The case $p = 1$ is special, because the sinc function is not in $L^1(\mathbb{R})$. Hence, it is not obvious how to find a sequence of elementary computable functions that approximates $f \in \mathcal{B}_\pi^1$ in the $\mathcal{B}_\pi^1$-norm. For $L \in \mathbb{N}$, consider, for example, the function

$$\begin{aligned} f_L(t) &= \sum_{k=-L}^{L} f(k) \frac{\sin(\pi(t-k))}{\pi(t-k)} \\ &\quad - \left(\sum_{k=-L}^{L} f(k)(-1)^k\right) \frac{\sin(\pi t)}{\pi t} \\ &= \sum_{k=-L}^{L} f(k)(-1)^k \sin(\pi t) \frac{k}{\pi(t-k)t}, \end{aligned} \tag{10}$$

where in the second equality we used that $\sin(\pi(t-k)) = (-1)^k \sin(\pi t)$ for all $t \in \mathbb{R}$, $k \in \mathbb{Z}$. Clearly, f_L is an elementary computable function, having the shape

$$\sum_{k=-L}^{L} \alpha_k \frac{\sin(\pi(t-k))}{\pi(t-k)},$$

with $\{\alpha_k\}_{k=-L}^{L}$ given by

$$\alpha_k = \begin{cases} f(0) - \sum_{k=-L}^{L} f(k)(-1)^k, & k = 0, \\ f(k), & \text{otherwise.} \end{cases}$$

Further, from (10) we see that $f_L \in \mathcal{B}_\pi^1$. Thus, $\{f_L\}_{L\in\mathbb{N}}$ is a computable sequence of elementary computable functions in $\mathcal{B}_\pi^1$. Note however, this sequence does not necessarily converge to f. For us in this paper this is not a problem. We are interested in the algorithmic computability of the actual bandwidth $B(f)$ for a suitable class of computable bandlimited signals. For $1 < p < \infty$ there are no problems when using elementary computable functions for the approximation. For $p = 1$, according to the definition of a computable signal in $\mathcal{B}_\pi^1$, we only consider those signals in $\mathcal{B}_\pi^1$ that can be effectively approximated by computable sequences of elementary computable functions. In other words, we restrict ourselves to signals that are "generated" by elementary computable functions. It could be mathematically interesting to consider larger classes of elementary functions, which would in turn lead to larger sets of computable signals. However, this is not our concern and even if doing so, our results would not be affected. We will show that the there exists a signal $f \in \mathcal{CB}_\pi^1$ such that $B(f)$ is not computable. This problem will clearly also exist for all potential extensions of the set $\mathcal{CB}_\pi^1$.

The following two facts are useful for the analysis of computability.

Fact 3. *Let $f \in \mathcal{CB}_\pi^1$. Then the Fourier transform $\hat{f}$ is a computable continuous function.*

The proof of Fact 3 will be given at the beginning of the proof of Theorem 4.

Fact 4. *Let $\{f_n\}_{n\in\mathbb{N}}$ be a computable sequence of computable functions in $\mathcal{B}_\pi^p$, $p \in [1,\infty) \cap \mathbb{R}_c$. If $\{f_n\}_{n\in\mathbb{N}}$ converges effectively to a limit f in the $\mathcal{B}_\pi^p$-norm, then we have $f \in \mathcal{CB}_\pi^p$, i.e., f is computable in $\mathcal{B}_\pi^p$.*

Proof. Since $\{f_n\}_{n\in\mathbb{N}}$ converges effectively to f in the $\mathcal{B}_\pi^p$-norm, there exists a recursive function $\xi\colon \mathbb{N} \to \mathbb{N}$ such that for all $N \in \mathbb{N}$ we have

$$\|f - f_n\|_{\mathcal{B}_\pi^p} \leq 2^{-N}$$

for all $n \geq \xi(N)$. For $N \in \mathbb{N}$ let $g_N = f_{\xi(N+1)}$. Then $\{g_N\}_{N\in\mathbb{N}}$ is a computable sequence of computable functions in $\mathcal{B}_\pi^p$ with

$$\|f - g_N\|_{\mathcal{B}_\pi^p} \leq \frac{1}{2^{N+1}}.$$

Moreover, since $\{g_N\}_{N\in\mathbb{N}}$ is computable sequence of computable functions in $\mathcal{B}_\pi^p$, we can find, using the same arguments, a computable double sequence of elementary computable functions $\{g_{N,m}\}_{N\in\mathbb{N},m\in\mathbb{N}}$ with

$$\|g_N - g_{N,m}\|_{\mathcal{B}_\pi^p} \leq \frac{1}{2^{m+1}}.$$

Hence, for $N \in \mathbb{N}$ and $m = N$ we have

$$\|g_N - g_{N,N}\|_{\mathcal{B}_\pi^p} \leq \frac{1}{2^{N+1}}.$$

It follows that

$$\begin{aligned}\|f - g_{N,N}\|_{\mathcal{B}_\pi^p} &\leq \|f - g_N\|_{\mathcal{B}_\pi^p} + \|g_N - g_{N,N}\|_{\mathcal{B}_\pi^p} \\ &\leq \frac{1}{2^{N+1}} + \frac{1}{2^{N+1}} \\ &= \frac{1}{2^N}\end{aligned}$$

for all $N \in \mathbb{N}$. Since $\{g_{N,N}\}_{N\in\mathbb{N}}$ is a computable sequence of elementary computable functions, the proof is complete. □

Finally, we also need to define computability for mappings that map computable signals into real numbers.

Definition 8. We say that a mapping $\Gamma\colon \mathcal{CB}_\pi^p \to \mathbb{R}_c$, $p \in [1,\infty) \cap \mathbb{R}_c$, is computable if there exists a Turing machine that, for every input $f \in \mathcal{CB}_\pi^p$, can compute $\Gamma(f)$.

V. Computability of the Minimum Bandwidth

In this section we analyze for signals in spaces $\mathcal{B}_\pi^1$ and $\mathcal{B}_\pi^2$ whether it is possible to determine their actual bandwidth algorithmically, i.e., we study Question 1 from the introduction. To this end, we need to restrict ourselves to computable signals, i.e., signals in the spaces $\mathcal{CB}_\pi^1$ and $\mathcal{CB}_\pi^2$. Hence, we rephrase the question.

- Question 1A: Does there exist an algorithm that, for every computable signal $f \in \mathcal{CB}_\pi^1$ (or $f \in \mathcal{CB}_\pi^2$), is able to compute $B(f)$?

This question is practically relevant, since the actual bandwidth $B(f)$ of a bandlimited signal f is an important quantity. For example, it is directly linked to the minimum sampling rate in various sampling theorems, for details see above.

A necessary condition for the existence of such an algorithm is that the mapping B maps computable functions into computable numbers. Hence, the next question is weaker than Question 1A.

- Question 1B: Do we have $B(f) \in \mathbb{R}_c$ for all $f \in \mathcal{CB}_\pi^1$ (or $f \in \mathcal{CB}_\pi^2$)?

Remark 4. The claim in Question 1B is indeed weaker than the claim in Question 1A. If the answer to Question 1B was "yes" then, for every signal f, $B(f)$ would be a computable number; and, being a computable number, there would exist some algorithm that computes the computable number $B(f)$ with arbitrary precision. However, this would not mean that we can find an algorithm that recursively depends on f, i.e., an algorithm that takes an arbitrary f as input and then computes $B(f)$. If the answer to Question 1A was "yes", then exactly this would be possible, i.e., we could find such an algorithm that recursively depends on f.

We first study Question 1B for signals $f \in \mathcal{CB}_\pi^2$. Then we study the same question for $f \in \mathcal{CB}_\pi^1$. In both cases we have to answer Question 1B, and consequently Question 1A in the negative.

Theorem 1. *There exists a signal $f_1 \in \mathcal{CB}_\pi^2$ such that $B(f_1) \notin \mathbb{R}_c$, i.e., $B(f_1)$ is not Turing computable.*

Remark 5. The fact that we have $B(f_1) \notin \mathbb{R}_c$ shows that the most basic requirement for the computability of a function—the property that computable objects are mapped into computable objects—is not satisfied. Hence, Question 1A has to be answered in the negative for $\mathcal{CB}_\pi^2$.

For the signal f_1, which we will construct in the proof of Theorem 1, we cannot compute any of the sampling points

$$\frac{k\pi}{B(f_1)}, \quad k \in \mathbb{Z}, k \neq 0.$$

If we could compute

$$\lambda = \frac{k_1\pi}{B(f_1)}$$

for some $k_1 \in \mathbb{Z}$, $k_1 \neq 0$, then we could compute

$$B(f_1) = \frac{k_1\pi}{\lambda},$$

because $\lambda \in \mathbb{R}_c$. But that would be a contradiction.

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Proof of Theorem 1. Let $\mathcal{A} \subsetneq \mathbb{N}$ be a recursively enumerable nonrecursive set, and $\phi_{\mathcal{A}} \colon \mathbb{N} \to \mathcal{A}$ a recursive enumeration of the elements of $\mathcal{A}$, where $\phi_{\mathcal{A}}$ is a one-to-one function, i.e., for every element $m \in \mathcal{A}$ there exists exactly one $k \in \mathbb{N}$ with $\phi_{\mathcal{A}}(k) = m$. For $l \in \mathbb{N}$, let

$$a_l = \sum_{k=1}^{l} \frac{1}{2^{\phi_{\mathcal{A}}(k)}}.$$

We have $a_l < a_{l+1} < 1$, $l \in \mathbb{N}$, and

$$\omega_* := \lim_{l\to\infty} a_l = \sum_{k=1}^{\infty} \frac{1}{2^{\phi_{\mathcal{A}}(k)}}.$$

is a non-computable real number, according to Lemma 1. We set

$$f_1(t) = \sum_{l=1}^{\infty} \frac{1}{2^l} \frac{\sin(a_l(t-l))}{\pi(t-l)}, \quad t \in \mathbb{R}.$$

The Fourier transform of f_1 is given by

$$\hat{f}_1(\omega) = \sum_{l=1}^{\infty} \frac{1}{2^l} \mathrm{e}^{-il\omega} \operatorname{rect}_{a_l}(\omega),$$

where

$$\operatorname{rect}_{a_l}(\omega) = \begin{cases} 1, & |\omega| < a_l, \\ 0, & |\omega| \geq a_l. \end{cases}$$

Further, we have

$$\begin{aligned} \|f_1\|_{\mathcal{B}_\pi^2} &\leq \sum_{l=1}^{\infty} \frac{1}{2^l} \left\| \frac{\sin(a_l(\,\cdot\,-l))}{\pi(\,\cdot\,-l)} \right\|_{\mathcal{B}_\pi^2} \\ &= \sum_{l=1}^{\infty} \frac{1}{2^l} \sqrt{\frac{a_l}{\pi}} \\ &< \sum_{l=1}^{\infty} \frac{1}{2^l} \\ &= 1 \\ &< \infty, \end{aligned}$$

where we used the fact that $a_l < 1$, $l \in \mathbb{N}$, in the second inequality. Since, as we show in Appendix D,

$$\frac{\sin(a_l(t-k))}{\pi(t-k)}$$

is a computable function in $\mathcal{B}_\pi^2$ for all $l \in \mathbb{N}$, and

$$\begin{aligned} \left\| f_1 - \sum_{l=1}^{M} \frac{1}{2^l} \frac{\sin(a_l(\,\cdot\,-l))}{\pi(\,\cdot\,-l)} \right\|_{\mathcal{B}_\pi^2} &\leq \sum_{l=M+1}^{\infty} \frac{1}{2^l} \sqrt{\frac{a_l}{\pi}} \\ &< \sum_{l=M+1}^{\infty} \frac{1}{2^l} \\ &= \frac{1}{2^{M+1}} \frac{1}{1-\frac{1}{2}} \\ &= \frac{1}{2^M}, \end{aligned}$$

we see that the computable sequence

$$\left\{ \sum_{l=1}^{M} \frac{1}{2^l} \frac{\sin(a_l(\,\cdot\,-l))}{\pi(\,\cdot\,-l)} \right\}_{M=1}^{\infty}$$

of computable functions in $\mathcal{B}_\pi^2$ converges effectively in the $\mathcal{B}_\pi^2$-norm, and hence that f_1 is computable in $\mathcal{B}_\pi^2$. For $\omega \in (-\infty, -\omega_*] \cup [\omega_*, \infty)$, we have $\hat{f}_1(\omega) = 0$ almost everywhere. Therefore, we see that

$$B(f_1) \leq \omega_*. \tag{11}$$

Let $k \in \mathbb{N}$ be arbitrary. For $\omega \in (a_k, a_{k+1})$ we have

$$\hat{f}_1(\omega) = \sum_{l=k+1}^{\infty} \frac{1}{2^l} \mathrm{e}^{-il\omega} = \frac{\mathrm{e}^{-i(k+1)\omega}}{2^{k+1}} \frac{1}{1-\frac{\mathrm{e}^{i\omega}}{2}},$$

which implies that $|\hat{f}_1(\omega)| > 0$ for all $\omega \in (a_k, a_{k+1})$. Thus, we see that $B(f_1) \geq a_{k+1}$. Since this is true for all $k \in \mathbb{N}$, it follows that

$$B(f_1) \geq \omega_*. \tag{12}$$

Combining (11) and (12), we obtain that $B(f_1) = \omega_*$, which implies that $B(f_1) \notin \mathbb{R}_c$. □

Next, we present an alternative proof, which is based on the construction of a different signal. While the signal f_1 in the previous proof had a simple time domain structure in the form of a Shannon sampling series, the signal f_2 in the following proof is particularly simple in the frequency domain.

Remark 6. For $t \in \mathbb{R}$, let

$$f_2(t) = \sum_{l=1}^{\infty} \frac{1}{l^2} \left(\frac{\sin(a_l(t-l))}{\pi(t-l)} - \frac{\sin(a_{l+1}(t-l))}{\pi(t-l)} \right). \tag{13}$$

Similar as before, it is shown that $f_2 \in \mathcal{B}_\pi^2$, and that the series in (13) converges effectively. This shows that $f_2 \in \mathcal{CB}_\pi^2$. For $\omega \in (-\infty, -\omega_*] \cap [\omega_*, \infty)$, we have $\hat{f}_2(\omega) = 0$ almost everywhere. Hence, we see that

$$B(f_2) \leq \omega_*. \tag{14}$$

Let $k \in \mathbb{N}$ be arbitrary. For $\omega \in (a_k, a_{k+1})$ we have

$$\hat{f}_2(\omega) = \frac{1}{k^2} \mathrm{e}^{-ik\omega},$$

which implies that $|\hat{f}_2(\omega)| > 0$ for all $\omega \in (a_k, a_{k+1})$. Thus, we see that $B(f_2) \geq a_{k+1}$. Since this is true for all $k \in \mathbb{N}$, it follows that

$$B(f_2) \geq \omega_*. \tag{15}$$

Combining (14) and (15), it follows that $B(f_2) = \omega_*$, and consequently that $B(f_2) \notin \mathbb{R}_c$.

The next result is similar to Theorem 1, however, the signal space is different. In the following theorem we consider computable signals in $\mathcal{B}_\pi^1$.

Theorem 2. *There exists a signal $f_3 \in \mathcal{CB}_\pi^1$ such that $B(f_3) \notin \mathbb{R}_c$, i.e., $B(f_3)$ is not Turing computable.*

Proof. Let $\mathcal{A} \subsetneq \mathbb{N}$ be a recursively enumerable nonrecursive set, and $\phi_{\mathcal{A}} \colon \mathbb{N} \to \mathcal{A}$ a recursive enumeration of the elements of $\mathcal{A}$. As in the proof of Theorem 1, we set

$$a_l = \sum_{k=1}^{l} \frac{1}{2^{\phi_{\mathcal{A}}(k)}}$$

and

$$\omega_* := \lim_{l\to\infty} a_l = \sum_{k=1}^{\infty} \frac{1}{2^{\phi_{\mathcal{A}}(k)}}.$$

For $\delta > 0$, let

$$\begin{aligned} g_\delta(t) &= \frac{1}{2\pi}\int_{-\delta}^{\delta}\left(1-\frac{|\omega|}{\delta}\right)\mathrm{e}^{i\omega t}\;\mathrm{d}\omega \\ &= \frac{\delta}{2\pi}\left(\frac{\sin(\frac{\delta t}{2})}{\frac{\delta t}{2}}\right)^2, \quad t\in\mathbb{R}. \end{aligned} \tag{16}$$

The second equality in (16) can be obtained by partial integration. A direct calculation shows that

$$\begin{aligned} \int_{-\infty}^{\infty} |g_\delta(t)|\;\mathrm{d}t &= \frac{\delta}{2\pi}\int_{-\infty}^{\infty}\left(\frac{\sin(\frac{\delta t}{2})}{\frac{\delta t}{2}}\right)^2\;\mathrm{d}t \\ &= \frac{\delta}{(2\pi)^2}\int_{-\delta/2}^{\delta/2}\left(\frac{2\pi}{\delta}\right)^2\;\mathrm{d}\omega \\ &= 1. \end{aligned} \tag{17}$$

For $l\in\mathbb{N}$, we set

$$\hat{q}_l(\omega) = \hat{g}_{\frac{a_{l+1}-a_l}{2}}(\omega - \frac{a_l+a_{l+1}}{2}),$$

or, equivalently, in the time domain

$$q_l(t) = g_{\frac{a_{l+1}-a_l}{2}}(t)\,\mathrm{e}^{i\frac{a_l+a_{l+1}}{2}t}.$$

In Appendix E we show that, for $\delta\in(0,\pi)\cap\mathbb{R}_c$, g_δ is a computable function in $\mathcal{B}_\pi^1$. Since $\{a_l\}_{l\in\mathbb{N}}$ is a computable sequence of rational numbers, it follows that $q_l\in\mathcal{CB}_\pi^1$ for all $l\in\mathbb{N}$, and that $\{q_l\}_{l\in\mathbb{N}}$ is a computable sequence of functions in $\mathcal{CB}_\pi^1$. Further, we have $q_l\in\mathcal{B}_{a_{l+1}}^1$ for all $l\in\mathbb{N}$. Let

$$f_3(t) = \sum_{l=1}^{\infty}\frac{1}{l^2}q_l(t), \quad t\in\mathbb{R}. \tag{18}$$

We have

$$\|f_3\|_{\mathcal{B}_\pi^1} \le \sum_{l=1}^{\infty}\frac{1}{l^2}\|q_l\|_{\mathcal{B}_\pi^1} = \sum_{l=1}^{\infty}\frac{1}{l^2} = \frac{\pi^2}{2},$$

where we used (17) in the last inequality. The series in (18) converges effectively in the $\mathcal{B}_\pi^1$-norm which shows that $f_3\in\mathcal{CB}_\pi^1$. Since $q_l\in\mathcal{B}_{\omega_*}^1$ for all $l\in\mathbb{N}$, it follows that $f_3\in\mathcal{B}_{\omega_*}^1$, i.e., that

$$B(f_3)\le\omega_*. \tag{19}$$

Let $k\in\mathbb{N}$ be arbitrary. For $\omega\in(a_k,a_{k+1})$ we have

$$|\hat{f}_3(\omega)| = \frac{1}{k^2}\hat{g}_{\frac{a_{k+1}-a_k}{2}}(\omega-\frac{a_k+a_{k+1}}{2}) > 0.$$

This implies that $B(f_3)\ge a_{k+1}$ for all $k\in\mathbb{N}$, and consequently that

$$B(f_3)\ge\omega_*. \tag{20}$$

Combining (19) and (20), it follows that $B(f_3)=\omega_*$, and consequently that $B(f_3)\notin\mathbb{R}_c$, according to Lemma 1. □

VI. Semi Decidability

We have seen in Theorems 1 and 2 that there exist signals f in $\mathcal{B}_\pi^2$ and in $\mathcal{B}_\pi^1$ such that $B(f)\notin\mathbb{R}_c$. Hence, for those signals, it is impossible to algorithmically compute $B(f)$—there exists no algorithm that can perform this task. This raises the question whether we can algorithmically detect the cases where this behavior occurs. It would be useful to have an algorithmic test that can determine whether, for a given signal f, we have $B(f)\in\mathbb{R}_c$, or not. Such an algorithmic test is desirable, for example, for computer aided signal parameter estimation. By means of a signal preselection, the computer could eliminate the non-admissible signals beforehand. In this section we show that such an algorithmic test cannot exist.

Definition 9. We call a set $\mathcal{M}\subseteq\mathcal{CB}_\pi^1$ *semi-decidable* if there exists a Turing machine

$$\mathrm{TM}\colon\mathcal{CB}_\pi^1\to\{\text{TM stops, TM runs forever}\}$$

that, given an input $f\in\mathcal{CB}_\pi^1$, stops if and only if $f\in\mathcal{M}$.

Remark 7. If $\mathcal{M}$ is semi-decidable, the Turing machine TM accepts exactly the elements of $\mathcal{M}$.

Let

$$\mathcal{C}_{\mathrm{BW}}^1 = \left\{f\in\mathcal{CB}_\pi^1\colon B(f)\in\mathbb{R}_c\right\}$$

denote the set of all signals in $\mathcal{CB}_\pi^1$ for which $B(f)$ can be computed algorithmically, and

$$\mathcal{NC}_{\mathrm{BW}}^1 = \mathcal{CB}_\pi^1\setminus\mathcal{C}_{\mathrm{BW}}^1 = \left\{f\in\mathcal{CB}_\pi^1\colon B(f)\notin\mathbb{R}_c\right\}.$$

the set of all signals in $\mathcal{CB}_\pi^1$, for which $B(f)$ cannot be computed algorithmically.

In view of our above discussion, it would be desirable to have a Turing machine that can decide whether $B(f)\in\mathbb{R}_c$ or $B(f)\notin\mathbb{R}_c$. We will show that such a Turing machine cannot exist, by proving that even the weaker problem of semi-decidability cannot be answered positively.

Theorem 3. *Neither $\mathcal{C}_{BW}^1$ nor $\mathcal{NC}_{BW}^1$ is semi-decidable.*

Proof. Let $\mathcal{A}\subsetneq\mathbb{N}$ be a recursively enumerable nonrecursive set, and $\phi_{\mathcal{A}}\colon\mathbb{N}\to\mathcal{A}$ a recursive enumeration of the elements of $\mathcal{A}$. As in the proof of Theorem 1, we set

$$a_l = \sum_{k=1}^{l}\frac{1}{2^{\phi_{\mathcal{A}}(k)}}$$

and

$$\omega_* := \lim_{l\to\infty} a_l = \sum_{k=1}^{\infty}\frac{1}{2^{\phi_{\mathcal{A}}(k)}}.$$

According to Lemma 1, we have $\omega_*\notin\mathbb{R}_c$. Further, let $f_3\in\mathcal{CB}_\pi^1$ be defined as in the proof of Theorem 2.

We first prove that $\mathcal{C}_{\mathrm{BW}}^1$ is not semi-decidable. Let

$$g(t) = \left(\frac{\sin(a_1 t/2)}{\pi t}\right)^2.$$

We have $B(g)=a_1\in\mathbb{R}_c$ and $g\in\mathcal{CB}_\pi^1$, as we show in Appendix E. For $\lambda\in[0,1]$, we set

$$g_\lambda(t) = g(t)+\lambda f_3(t), \quad t\in\mathbb{R}.$$

Then we have $g_0 = g \in \mathcal{C}^1_{\text{BW}}$. Further, we have $g_\lambda \in \mathcal{CB}^1_\pi$ for all $\lambda \in [0,1] \cap \mathbb{R}_c$. Note that we can effectively approximate g_λ, independently of λ. To see this, let $\{g_N\}_{N\in\mathbb{N}}$ be a sequence of elementary computable functions that satisfies

$$\|g - g_N\|_{\mathcal{B}^1_\pi} \leq \frac{1}{2^{N+1}}, \qquad N \in \mathbb{N},$$

and $\{f_{3,N}\}_{N\in\mathbb{N}}$ a sequence of elementary computable functions that satisfies

$$\|f_3 - f_{3,N}\|_{\mathcal{B}^1_\pi} \leq \frac{1}{2^{N+1}}, \qquad N \in \mathbb{N},$$

and let

$$u_{\lambda,N}(t) = g_N(t) + \lambda f_{3,N}(t), \quad t \in \mathbb{R}.$$

Then, for all $\lambda \in [0,1] \cap \mathbb{R}_c$, we have

$$\begin{aligned}\|g_\lambda - u_{\lambda,N}\|_{\mathcal{B}^1_\pi} &\leq \|g - g_N + \lambda f_3 - \lambda f_{3,N}\|_{\mathcal{B}^1_\pi} \\ &\leq \|g - g_N\|_{\mathcal{B}^1_\pi} + \lambda \|f_3 - f_{3,N}\|_{\mathcal{B}^1_\pi} \\ &\leq \frac{1}{2^{N+1}} + \lambda \frac{1}{2^{N+1}} \\ &\leq \frac{1}{2^N}.\end{aligned}$$

For $|\omega| > a_1$ we have

$$\hat{g}_\lambda(\omega) = \lambda \hat{f}_3(\omega).$$

Hence, we see that $B(g_\lambda) = \omega_* \notin \mathbb{R}_c$, i.e. $g_\lambda \in \mathcal{NC}^1_{\text{BW}}$ for all $\lambda \in (0,1] \cap \mathbb{R}_c$. We use a proof by contradiction and assume that $\mathcal{C}^1_{\text{BW}}$ is semi-decidable. Since $\mathcal{C}^1_{\text{BW}}$ is semi-decidable, there exists a Turing machine $\text{TM}_{\mathcal{C}^1_{\text{BW}}}$ that stops if and only if $f \in \mathcal{C}^1_{\text{BW}}$. Further, we use the fact that there exists a Turing machine $\text{TM}_>$ that, given an input $\lambda \in \mathbb{R}_c$, stops if and only if $\lambda > 0$ [26, Proposition 0, p. 14]. Using those two Turing machines, we build a new Turing machine according to

$$\text{TM}(\lambda) = \begin{cases} 0, & \text{TM}_{\mathcal{C}^1_{\text{BW}}}(g_\lambda) \text{ stops,} \\ 1, & \text{TM}_>(\lambda) \text{ stops.} \end{cases}$$

TM is a Turing machine that can decide for $\lambda \in [0,1] \cap \mathbb{R}_c$ whether $\lambda = 0$ or $\lambda > 0$. This is a contradiction, because such a Turing machine cannot exist [26, Proposition 0, p. 14].

Next, we prove that $\mathcal{NC}^1_{\text{BW}}$ is not semi-decidable. For $\lambda \in [0,1]$ let

$$\phi_\lambda(t) = f_3(t) + \lambda \left(\frac{\sin(\pi t/2)}{\pi t}\right)^2, \quad t \in \mathbb{R}.$$

For $\lambda \in [0,1] \cap \mathbb{R}_c$ we have $\phi_\lambda \in \mathcal{CB}^1_\pi$. For $\lambda = 0$ we have $\phi_0 = f_3 \in \mathcal{NC}^1_{\text{BW}}$, because $B(\phi_0) = B(f_3) = \omega_* \notin \mathbb{R}_c$. For $\lambda \in (0,1] \cap \mathbb{R}_c$ we have $B(\phi_\lambda) = \pi \in \mathbb{R}_c$ and therefore $\phi_\lambda \in \mathcal{C}^1_{\text{BW}}$. Again, we use a proof by contradiction. We assume that $\mathcal{NC}^1_{\text{BW}}$ is semi-decidable and derive a contradiction. Since $\mathcal{NC}^1_{\text{BW}}$ is semi-decidable, there exists a Turing machine $\text{TM}_{\mathcal{NC}^1_{\text{BW}}}$ that stops if and only if $f \in \mathcal{NC}^1_{\text{BW}}$. We consider the Turing machine

$$\text{TM}(\lambda) = \begin{cases} 0, & \text{TM}_{\mathcal{NC}^1_{\text{BW}}}(\phi_\lambda) \text{ stops,} \\ 1, & \text{TM}_>(\lambda) \text{ stops.} \end{cases}$$

TM is a Turing machine that can decide for $\lambda \in [0,1] \cap \mathbb{R}_c$ whether $\lambda = 0$ or $\lambda > 0$. This is a contradiction, because such a Turing machine cannot exist [26, Proposition 0, p. 14]. □

Remark 8. The theory for $\mathcal{B}^2_\pi$ is the same, i.e., neither

$$\mathcal{C}^2_{\text{BW}} = \left\{ f \in \mathcal{CB}^2_\pi \colon B(f) \in \mathbb{R}_c \right\}$$

nor

$$\mathcal{NC}^2_{\text{BW}} = \mathcal{CB}^2_\pi \setminus \mathcal{C}^2_{\text{BW}} = \left\{ f \in \mathcal{CB}^2_\pi \colon B(f) \notin \mathbb{R}_c \right\}$$

is semi-decidable.

VII. Approximate Bandwidth

In Section V we have seen that, in general, it is not possible to algorithmically compute the actual bandwidth $B(f)$ of a signal f. Next, we treat the Question 2 from the introduction. For a given number $\sigma > 0$, we want to algorithmically determine if $B(f) > \sigma$. The following theorem shows that this is possible, in the sense that there exists an algorithm that stops if and only if $B(f) > \sigma$.

For $\sigma \in (0,\pi) \cap \mathbb{R}_c$, let

$$\mathcal{C}^1_>(\sigma) = \left\{ f \in \mathcal{CB}^1_\pi \colon B(f) > \sigma \right\}.$$

Theorem 4. *For all $\sigma \in (0,\pi) \cap \mathbb{R}_c$ the set $\mathcal{C}^1_>(\sigma)$ is semi-decidable.*

According to this theorem, we can specify a bandwidth $\sigma > 0$ and find a Turing machine that stops if and only if the actual bandwidth $B(f)$ of the signal f is larger than σ. Unfortunately, this result does not allow us to determine an effective upper bound for $B(f)$, because this Turing machine does not stop if $B(f) \leq \sigma$.

If the Turing machine stops for a given signal $f \in \mathcal{CB}^1_\pi$ then it follows that f cannot be reconstructed from the samples $\{f(k\pi/\sigma)\}_{k\in\mathbb{Z}}$, in general, because the sampling rate σ/π is too low.

Proof of Theorem 4. Let $\sigma \in (0,\pi) \cap \mathbb{R}_c$ be arbitrary but fixed. For $f \in \mathcal{CB}^1_\pi$ we will construct a Turing machine $\text{TM}_{\mathcal{C}^1_>(\sigma)}$ that stops if and only if $f \in \mathcal{C}^1_>(\sigma)$. We have

$$\left| \hat{f}(\omega) - \int_{-n}^{n} f(t)\, \mathrm{e}^{-i\omega t}\, \mathrm{d}t \right| \leq \int_{|t|\geq n} |f(t)|\, \mathrm{d}t$$

for all $\omega \in [-\pi,\pi]$. Since $f \in \mathcal{CB}^1_\pi$, we can find a recursive function $\xi \colon \mathbb{N} \to \mathbb{N}$ such that, for all $N \in \mathbb{N}$, we have

$$\int_{|t|\geq n} |f(t)|\, \mathrm{d}t \leq \frac{1}{2^N}$$

for all $n \geq \xi(N)$. Thus, we see that $\hat{f}$ is a computable continuous function. In particular, $\hat{f}$ is computable on the intervals $[-\pi,-\sigma]$ and $[\sigma,\pi]$, where the endpoints of the intervals are computable numbers, and therefore the numbers

$$C_-(f) = \max_{\omega\in[-\pi,-\sigma]} |\hat{f}(\omega)|$$

and

$$C_+(f) = \max_{\omega\in[\sigma,\pi]} |\hat{f}(\omega)|$$

are computable real numbers [26, Theorem 7, p. 40]. Hence, there exist Turing machines TM_- and TM_+ that, for an input $f \in \mathcal{CB}_\pi^1$, compute $C_-(f)$ and $C_+(f)$, respectively. Further, we use that fact that there exists a Turing machine $\mathrm{TM}_>$ that, given an input $\lambda \in \mathbb{R}_c$, stops if and only if $\lambda > 0$. The concatenations

$$\overline{\mathrm{TM}}_-(f) = \mathrm{TM}_>(\mathrm{TM}_-(f))$$

and

$$\overline{\mathrm{TM}}_+(f) = \mathrm{TM}_>(\mathrm{TM}_+(f))$$

define two new Turing machines $\overline{\mathrm{TM}}_-$ and $\overline{\mathrm{TM}}_+$. $\overline{\mathrm{TM}}_-$ stops if and only if $C_-(f) > 0$ and $\overline{\mathrm{TM}}_+$ stops if and only if $C_+(f) > 0$. The Turing machine $\mathrm{TM}_{\mathcal{C}_>^1(\sigma)}$ that stops if $\overline{\mathrm{TM}}_-$ or $\overline{\mathrm{TM}}_+$ stops, and otherwise runs forever, is the desired Turing machine. If $f \in \mathcal{C}_>^1(\sigma)$, i.e. if $B(f) > \sigma$, then we have $C_-(f) > 0$ or $C_+(f) > 0$, and consequently $\mathrm{TM}_{\mathcal{C}_>^1(\sigma)}(f)$ stops. If $f \notin \mathcal{C}_>^1(\sigma)$, i.e. if $B(f) \le \sigma$, then we have $C_-(f) = 0$ and $C_+(f) = 0$, and consequently $\mathrm{TM}_{\mathcal{C}_>^1(\sigma)}(f)$ runs forever. □

The previous theorem can be extended to hold for the larger signal spaces $\mathcal{CB}_\pi^p$, $p \in (1,\infty) \cap \mathbb{R}_c$, and for $\mathcal{CB}_{\pi,0}^\infty$. For $\sigma \in (0,\pi) \cap \mathbb{R}_c$ and $p \in [1,\infty] \cap \mathbb{R}_c$, let

$$\mathcal{C}_>^p(\sigma) = \begin{cases} \{f \in \mathcal{CB}_\pi^p \colon B(f) > \sigma\}, & 1 \le p < \infty, \\ \{f \in \mathcal{CB}_{\pi,0}^\infty \colon B(f) > \sigma\}, & p = \infty. \end{cases}$$

For these sets, we have the following result.

Theorem 5. *Let $\sigma \in (0,\pi) \cap \mathbb{R}_c$ and $p \in [1,\infty] \cap \mathbb{R}_c$. Then the set $\mathcal{C}_>^p(\sigma)$ is semi-decidable.*

Proof. Let $\sigma \in (0,\pi) \cap \mathbb{R}_c$ and $p \in [1,\infty] \cap \mathbb{R}_c$ be arbitrary but fixed. Further, let $f \in \mathcal{CB}_\pi^p$ if $p < \infty$ and $f \in \mathcal{CB}_{\pi,0}^\infty$ if $p = \infty$. Let

$$g(z) = \frac{\left(f(\frac{z}{2}) - f(0)\right)^2}{z^2}, \quad z \in \mathbb{C}.$$

At $z = 0$ the function g has a removable singularity, and we have $g \in L^1(\mathbb{R})$. For all $\epsilon > 0$ there exists a constant $C_1(\epsilon)$ such that for all $|z| \ge 1$ we have

$$\begin{aligned} |g(z)| &= \frac{|f(\frac{z}{2}) - f(0)|^2}{|z|^2} \\ &\le |f(\tfrac{z}{2})|^2 + 2|f(0)||f(\tfrac{z}{2})| + |f(0)|^2 \\ &\le C_1(\epsilon)\,\mathrm{e}^{(B(f)+\epsilon)|z|}. \end{aligned} \tag{21}$$

In the second inequality of (21) we used that $f(z/2)^2$ is a bandlimited signal with bandwidth $B(f)$. Since (21) is valid for all $\epsilon > 0$, we see that $B(g) \le B(f)$. Next, we will show that $B(g) \ge B(f)$, which then implies that $B(g) = B(f)$. Since

$$|g(z)| \cdot |z|^2 = |f(\tfrac{z}{2}) - f(0)|^2,$$

it follows that

$$\sqrt{|g(z)|} \cdot |z| = |f(\tfrac{z}{2}) - f(0)| \ge |f(\tfrac{z}{2})| - |f(0)|. \tag{22}$$

Let $\mu > 0$ be arbitrary. There exists a $\hat{z} = \hat{z}(\mu)$ such that for all $|z| \ge \hat{z}(\mu)$ we have

$$\mathrm{e}^{\mu|z|} \ge |z|. \tag{23}$$

Combining (22), (21), and (23), we see that

$$\sqrt{C_1(\epsilon)}\,\mathrm{e}^{(\frac{\epsilon}{2}+\mu+\frac{B(g)}{2})|z|} \ge |f(\tfrac{z}{2})| - |f(0)| \tag{24}$$

for all $|z| \ge \hat{z}(\mu)$. Let $\delta > 0$ with $\delta < B(f)$ be arbitrary but fixed. For all $K \in \mathbb{N}$ there exists a $z_K \in \mathbb{C}$ such that

$$|f(z_K)| \ge K\,\mathrm{e}^{(B(f))-\delta)|z_K|}.$$

Clearly, we have

$$\lim_{K\to\infty} |z_K| = \infty,$$

because for all $R > 0$ there exists a $\overline{C}(R)$ with

$$|f(z)| \le \overline{C}(R)$$

for all $|z| \le R$. For K large enough such that $|z_K| > \hat{z}(\mu)$ we have

$$|f(\tfrac{z_K}{2})| \ge K\,\mathrm{e}^{(\frac{(B(f))}{2}-\frac{\delta}{2})|z_K|},$$

which, together with (24), gives

$$\sqrt{C_1(\epsilon)}\,\mathrm{e}^{(\frac{\epsilon}{2}+\mu+\frac{B(g)}{2}-\frac{B(f)}{2}+\frac{\delta}{2})|z_K|} \ge K - \frac{|f(0)|}{\mathrm{e}^{(\frac{B(f)}{2}-\frac{\delta}{2})|z_K|}}. \tag{25}$$

Since (25) holds for all sufficiently large K, it follows that

$$\frac{\epsilon}{2} + \mu + \frac{B(g)}{2} - \frac{B(f)}{2} + \frac{\delta}{2} > 0,$$

or, equivalently, that

$$\epsilon + 2\mu + B(g) + \delta > B(f).$$

Since $\epsilon > 0$, $\mu > 0$, and $\delta > 0$ were arbitrary, it follows that $B(g) \ge B(f)$. Hence, we have $g \in \mathcal{B}_\pi^1$ with $B(g) = B(f)$. In Appendix VII we will prove that g is computable in $\mathcal{B}_\pi^1$, i.e., that $g \in \mathcal{CB}_\pi^1$. To complete the proof, we use the Turing machine $\mathrm{TM}_{\mathcal{C}_>^1(\sigma)}$ from the proof of Theorem 4. $\mathrm{TM}_{\mathcal{C}_>^1(\sigma)}(g)$ stops if and only if $B(f) = B(g) > \sigma$. Consequently, $\mathcal{C}_>^p(\sigma)$ is semi-decidable. □

Next, we show that, for $\sigma \in (0,\pi) \cap \mathbb{R}_c$, the set

$$\mathcal{C}_\le^1(\sigma) = \{f \in \mathcal{CB}_\pi^1 \colon B(f) \le \sigma\}$$

is not semi-decidable.

Theorem 6. *For all $\sigma \in (0,\pi) \cap \mathbb{R}_c$ the set $\mathcal{C}_\le^1(\sigma)$ is not semi-decidable.*

Proof. We use a proof by contradiction. Assume that there exists a $\sigma \in (0,\pi) \cap \mathbb{R}_c$ such that $\mathcal{C}_\le^1(\sigma)$ is semi-decidable. Then there exists a Turing machine $\mathrm{TM}_{\mathcal{C}_\le^1(\sigma)}$ that stops for $f \in \mathcal{CB}_\pi^1$ if and only if $f \in \mathcal{C}_\le^1(\sigma)$. We consider the two functions

$$f_0(t) = \left(\frac{\sin\left(\frac{\sigma t}{2}\right)}{\sigma t}\right)^2$$

and

$$f_1(t) = \left(\frac{\sin\left(\frac{\pi t}{2}\right)}{\pi t}\right)^2.$$

We have $f_0, f_1 \in \mathcal{CB}_\pi^1$, which can be shown by the same calculation as in Appendix E, as well as $B(f_0) = \sigma$ and $B(f_1) = \pi$. For $\lambda \in [0,1] \cap \mathbb{R}_c$ let

$$f_\lambda(t) = (1-\lambda) f_0(t) + \lambda f_1(t), \quad t \in \mathbb{R}.$$

Then we have $f_\lambda \in \mathcal{CB}_\pi^1$ for all $\lambda \in [0,1] \cap \mathbb{R}_c$. Note that we can effectively approximate f_λ, independently of λ. To see this, let $\{g_{0,N}\}_{N\in\mathbb{N}}$ be a sequence of elementary computable functions that satisfies

$$\|f_0 - g_{0,N}\|_{\mathcal{B}_\pi^1} \leq \frac{1}{2^N}, \qquad N \in \mathbb{N},$$

and $\{g_{1,N}\}_{N\in\mathbb{N}}$ a sequence of elementary computable functions that satisfies

$$\|f_1 - g_{1,N}\|_{\mathcal{B}_\pi^1} \leq \frac{1}{2^N}, \qquad N \in \mathbb{N},$$

and let

$$g_{\lambda,N}(t) = (1-\lambda) g_{0,N}(t) + \lambda g_{1,N}(t), \quad t \in \mathbb{R}.$$

Then, for all $\lambda \in [0,1] \cap \mathbb{R}_c$, we have

$$\begin{aligned}
&\|f_\lambda - g_{\lambda,N}\|_{\mathcal{B}_\pi^1} \\
&\quad \leq \|(1-\lambda) f_0 - (1-\lambda) g_{0,N} + \lambda f_1 - \lambda g_{1,N}\|_{\mathcal{B}_\pi^1} \\
&\quad \leq (1-\lambda)\|f_0 - g_{0,N}\|_{\mathcal{B}_\pi^1} + \lambda \|f_1 - g_{1,N}\|_{\mathcal{B}_\pi^1} \\
&\quad \leq (1-\lambda)\frac{1}{2^N} + \lambda \frac{1}{2^N} \\
&\quad = \frac{1}{2^N}.
\end{aligned}$$

Further, we have $B(f_\lambda) = \pi$ for $\lambda \in (0,1] \cap \mathbb{R}_c$ and $B(f_\lambda) = \sigma$ for $\lambda = 0$. We consider the Turing machine

$$\mathrm{TM}(\lambda) = \begin{cases} 0, & \mathrm{TM}_{\mathcal{C}_\leq^1(\sigma)}(f_\lambda) \text{ stops}, \\ 1, & \mathrm{TM}_{\mathcal{C}_>^1(\sigma)}(f_\lambda) \text{ stops}, \end{cases}$$

where $\mathrm{TM}_{\mathcal{C}_>^1(\sigma)}$ is the Turing machine from the proof of Theorem 4. Since $\mathrm{TM}_{\mathcal{C}_\leq^1(\sigma)}(f_\lambda)$ stops if and only if $B(f_\lambda) \leq \sigma$ and $\mathrm{TM}_{\mathcal{C}_>^1(\sigma)}(f_\lambda)$ stops if and only if $B(f_\lambda) > \sigma$, we see that $\mathrm{TM}(f_\lambda) = 0$ if and only of $\lambda = 0$ and $\mathrm{TM}(f_\lambda) = 1$ if and only if $\lambda \in (0,1] \cap \mathbb{R}_c$. Hence, TM is a Turing machine that can decide for $\lambda \in [0,1] \cap \mathbb{R}_c$ whether $\lambda = 0$ or $\lambda > 0$. This is a contradiction, because such a Turing machine cannot exist [26, Proposition 0, p. 14]. □

Theorem 6 implies that for a given $\sigma \in (0,\pi) \cap \mathbb{R}_c$, we cannot determine algorithmically for all $f \in \mathcal{CB}_\pi^1$ whether f is uniquely determined by the samples $\{f(k\pi/\sigma)\}_{k\in\mathbb{Z}}$. We can only determine algorithmically the situation when the sequence of samples $\{f(k\pi/\sigma)\}_{k\in\mathbb{Z}}$ does not contain enough information for a unique reconstruction of f.

VIII. Approximate Bandwidth II

For a given number $\sigma > 0$ and a given signal $f \in \mathcal{CB}_\pi^1$ we would like to be able to algorithmically determine whether $B(f) < \sigma$. To study whether this is always possible we consider the set

$$\mathcal{C}_<^1(\sigma) = \left\{ f \in \mathcal{CB}_\pi^1 \colon B(f) < \sigma \right\}$$

and study whether $\mathcal{C}_<^1(\sigma)$ is semi-decidable.

In Theorem 6 in Section VII, we have already shown that, for all $\sigma \in (0,1) \cap \mathbb{R}_c$, the set

$$\mathcal{C}_\leq^1(\sigma) = \left\{ f \in \mathcal{CB}_\pi^1 \colon B(f) \leq \sigma \right\}$$

is not semi-decidable. For these sets, the non-semi-decidability could be caused by the condition $B(f) \leq \sigma$, in which σ is a sharp upper bound for the actual bandwidth. That is, σ/π is a sharp upper bound for the necessary sampling rate. Instead, for $f \in \mathcal{C}_<^1(\sigma)$, σ only gives a sufficient sampling rate σ/π, which always corresponds to oversampling.

However, even this modified question cannot be answered algorithmically in general.

Theorem 7. *There exist an $n \in \mathbb{N}$ and an $l \in \mathbb{N}$ with $1 \leq l \leq 2^n - 1$ such that $\mathcal{C}_<^1(l\pi/2^n)$ is not semi-decidable.*

In the proof of Theorem 7 we will use the signal $f_1 \in \mathcal{B}_\pi^1$ with $B(f_1) = \omega_* \notin \mathbb{R}_c$ from the proof of Theorem 1 to show that the set $\mathcal{C}_<^1(\sigma)$ is not always semi-decidable.

Proof of Theorem 7. We use a proof by contradiction. Assume that for all $n \in \mathbb{N}$ and all $l \in \mathbb{N}$ with $1 \leq l \leq 2^n - 1$ the set $\mathcal{C}_<^1(l\pi/2^n)$ is semi-decidable. Hence, for each $n \in \mathbb{N}$ and $1 \leq l \leq 2^n - 1$, there exists a Turing machine $\mathrm{TM}_{\mathcal{C}_<^1(l\pi/2^n)}$ that stops if and only if $f \in \mathcal{C}_<^1(l\pi/2^n)$. We already know from Theorem 4 that $\mathcal{C}_>^1(l\pi/2^n)$ is semi-decidable for all $1 \leq l \leq 2^n - 1$. For $n \in \mathbb{N}$ and $1 \leq l \leq 2^n - 1$, we denote by $\mathrm{TM}_{\mathcal{C}_>^1(l\pi/2^n)}$ the Turing machine that stops if and only if $f \in \mathcal{C}_>^1(l\pi/2^n)$. Let $f_1 \in \mathcal{CB}_\pi^1$ be the signal from Theorem 1 with $B(f_1) = \omega_* \notin \mathbb{R}_c$. Since ω_* is a non-computable real number, we have for all $1 \leq l \leq 2^n - 1$ that $\omega_* \neq l\pi/2^n$. Hence, we have for $n \in \mathbb{N}$ and $1 \leq l \leq 2^n - 1$ always either $f_1 \in \mathcal{C}_<^1(l\pi/2^n)$ or $f_1 \in \mathcal{C}_>^1(l\pi/2^n)$.

Let $n \in \mathbb{N}$ be arbitrary but fixed. On a universal Turing machine we start $2^n - 1$ Turing machines $\mathrm{TM}_{\mathcal{C}_>^1(l\pi/2^n)}(f_1)$, $1 \leq l \leq 2^n - 1$, and $2^n - 1$ Turing machines $\mathrm{TM}_{\mathcal{C}_<^1(l\pi/2^n)}(f_1)$, $1 \leq l \leq 2^n - 1$. Exactly $2^n - 1$ of these $2(2^n - 1)$ Turing machines stop. We wait until this has happened. Let $l_n^>$ be the largest index such that all Turing machines $\mathrm{TM}_{\mathcal{C}_>^1(l\pi/2^n)}(f_1)$ with $1 \leq l \leq l_n^>$ stopped. Further, let $l_n^<$ be the smallest index such that all Turing machines $\mathrm{TM}_{\mathcal{C}_<^1(l\pi/2^n)}(f_1)$ with $l_n^< \leq l \leq 2^n - 1$ stopped. Obviously, we have $l_n^< = l_n^> + 1$. Further, it is clear that if $\mathrm{TM}_{\mathcal{C}_<^1(\hat{l}\pi/2^n)}(f_1)$ stops for some $\hat{l}$, then we have $B(f_1) < \hat{l}\pi/2^n$, and, consequently, $\mathrm{TM}_{\mathcal{C}_<^1(l\pi/2^n)}(f_1)$ stops for all $\hat{l} \leq l \leq 2^n - 1$. Similarly, if $\mathrm{TM}_{\mathcal{C}_>^1(\hat{l}\pi/2^n)}(f_1)$ stops for some $\hat{l}$, then $\mathrm{TM}_{\mathcal{C}_>^1(l\pi/2^n)}(f_1)$ stops for all $1 \leq l \leq \hat{l}$. We have seen that both numbers $l_n^<$ and $l_n^>$ can be determined algorithmically by the universal Turing machine. We have

$$\frac{l_n^> \pi}{2^n} < \omega_* < \frac{l_n^< \pi}{2^n} = \frac{(l_n^> + 1)\pi}{2^n}.$$

Let

$$\lambda_n = \frac{l_n^> \pi}{2^n}.$$

Note that $\lambda_n \in \mathbb{R}_c$ and that we have

$$|\omega_* - \lambda_n| < \frac{\pi}{2^n}.$$

Since, $n \in \mathbb{N}$ was arbitrary, this procedure defines a computable sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ of computable numbers that converges effectively to ω_*. According to [26, Proposition 1, p. 20], this implies that $\omega_* \in \mathbb{R}_c$, which is a contradiction. □

The actual bandwidth $B(f_1) = B(f_3) = \omega_*$ of the computable signals f_1 and f_3 from Theorems 1 and 2, respectively, is not Turing computable. This number has the property that it is the limit of a monotonically increasing computable sequence of rational numbers. Next, we will show that all bandwidths $B(f)$ of signals $f \in \mathcal{CB}^p_\pi$, $p \in [1,\infty) \cap \mathbb{R}_c$, or $f \in \mathcal{CB}^\infty_{\pi,0}$ have this property. That is, we can always approximate $B(f)$ monotonically, however, the convergence is not effective in general, because $B(f)$ is not necessarily a computable number. This will be the statement of Theorem 8. In Theorem 9 we will show that if $B(f) \in \mathbb{R}_c$, then the sequence that is constructed in Theorem 8 converges effectively to $B(f)$.

Theorem 8. *For all $f \in \mathcal{CB}^\infty_{\pi,0}$ there exists a strictly monotonically increasing computable sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ of dyadic rational numbers such that $\lim_{n\to\infty} \lambda_n = B(f)$.*

Proof. Let $f \in \mathcal{CB}^\infty_{\pi,0}$ be arbitrary but fixed. The construction of the sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ is done similarly to the procedure in the proof of Theorem 7. From Theorem 5 we know that, for all $\sigma \in (0,\pi) \cap \mathbb{R}_c$, the set $\left\{f \in \mathcal{CB}^\infty_{\pi,0} \colon B(f) > \sigma\right\}$ is semi-decidable. Hence, for all $\sigma \in (0,\pi) \cap \mathbb{R}_c$ there exists a Turing machine $\mathrm{TM}_{\mathcal{C}^\infty_>(\sigma)}$ such that $\mathrm{TM}_{\mathcal{C}^\infty_>(\sigma)}(f)$ stops if and only if $B(f) > \sigma$.

We construct the sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ iteratively. In the first step, we start the Turing machine $\mathrm{TM}_{\mathcal{C}^\infty_>(\pi/2)}(f)$. We let this Turing machine run. In the second step, we start the three Turing machines $\mathrm{TM}_{\mathcal{C}^\infty_>(l\pi/2^2)}(f)$, $1 \leq l \leq 2^2 - 1$. We let these Turing machines run. Generally, in the jth step, we start $2^j - 1$ Turing machines $\mathrm{TM}_{\mathcal{C}^\infty_>(l\pi/2^j)}(f)$, $1 \leq l \leq 2^j - 1$, and let these run. This procedure is continued ad infinitum.

Since $\mathrm{TM}_{\mathcal{C}^\infty_>(\sigma)}(f)$ stops if and only if $B(f) > \sigma$, we know that eventually all Turing machines with $l\pi/2^j < B(f)$ will stop. We wait until one of the Turing machine stops. Let j_1 denote the iteration step in which this machine was started and l_1 the number of this machine. We set

$$\lambda_1 = \frac{l_1 \pi}{2^{j_1}}.$$

Now we wait until the second Turing machine stops. Let j_2 denote the iteration step in which this machine was started and l_2 the number of this machine. If

$$\frac{l_2 \pi}{2^{j_2}} > \lambda_1$$

then we set

$$\lambda_2 = \frac{l_2 \pi}{2^{j_2}}.$$

Then we wait until the next Turing machine stops, and so on. Using this procedure we generate a computable sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ with $\lambda_{n+1} > \lambda_n$, $n \in \mathbb{N}$. □

Theorem 9. *Let $f \in \mathcal{CB}^p_\pi$, $p \in [1,\infty) \cap \mathbb{R}_c$, or $f \in \mathcal{CB}^\infty_{\pi,0}$. If $B(f) \in \mathbb{R}_c$ then the sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ from the proof of Theorem 8 converges effectively to $B(f)$.*

Proof. Let $\{\lambda_n\}_{n\in\mathbb{N}}$ be the strictly monotonically increasing computable sequence from Theorem 8, and let $B(f) \in \mathbb{R}_c$. We first show that there exists a computable sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ of rational numbers with $\alpha_{n+1} \leq \alpha_n$, $n \in \mathbb{N}$, such that

$$|B(f) - \alpha_n| \leq \frac{1}{2^n},$$

i.e., the sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ converges effectively to $B(f)$. Since $B(f) \in \mathbb{R}_c$, there exists a computable sequence $\{\gamma_n\}_{n\in\mathbb{N}}$ of rational numbers such that

$$|B(f) - \gamma_n| \leq \frac{1}{2^n} \tag{26}$$

for all $n \in \mathbb{N}$. Hence, we have

$$B(f) \leq \gamma_n + \frac{1}{2^n}$$

for all $n \in \mathbb{N}$. Let

$$\alpha_n = \min_{1 \leq l \leq n} \left(\gamma_l + \frac{1}{2^l}\right), \quad n \in \mathbb{N}.$$

$\{\alpha_n\}_{n\in\mathbb{N}}$ is a computable sequence of rational numbers, and we have $\alpha_n \geq \alpha_{n+1}$ for all $n \in \mathbb{N}$. It follows that

$$|B(f) - \alpha_n| = \alpha_n - B(f) \leq \gamma_n + \frac{1}{2^n} - B(f) \leq \frac{2}{2^n}$$

for all $n \in \mathbb{N}$, where we used (26) in the last inequality. This shows that $\{\alpha_n\}_{n\in\mathbb{N}}$ converges effectively to $B(f)$.

Let

$$\delta_n = \alpha_n - \lambda_n, \quad n \in \mathbb{N}.$$

$\{\delta_n\}_{n\in\mathbb{N}}$ is a computable sequence of rational numbers, and we have

$$\delta_n = \alpha_n - \lambda_n \geq \alpha_{n+1} - \lambda_n \geq \alpha_{n+1} - \lambda_{n+1} = \delta_{n+1}$$

for all $n \in \mathbb{N}$. Thus, we see that $\{\delta_n\}_{n\in\mathbb{N}}$ is monotonically decreasing computable sequence of rational numbers with

$$\lim_{n\to\infty} \delta_n = 0.$$

For $N \in \mathbb{N}$, let $\xi(N)$ be the smallest natural number such that

$$\delta_{\xi(N)} < \frac{1}{2^N}.$$

Then we have

$$\delta_n < \frac{1}{2^N}$$

for all $n \geq \xi(N)$. It follows that

$$\alpha_n - \lambda_n < \frac{1}{2^N}$$

for all $n \geq \xi(N)$. For arbitrary $l > n$ we have

$$\alpha_l - \lambda_n < \frac{1}{2^N}$$

because $\alpha_l \leq \alpha_n$. Hence, we have

$$|B(f) - \lambda_n|B(f) - \lambda_n = \lim_{l\to\infty} \alpha_l - \lambda_n < \frac{1}{2^N}$$

for all $n \geq \xi(N)$, i.e., we have effective convergence of the sequence $\{\lambda_n\}_{n\in\mathbb{N}}$. □

The next theorem shows that for $f \in \mathcal{CB}^p_\pi$, $p \in [1,\infty) \cap \mathbb{R}_c$, and for $f \in \mathcal{CB}^\infty_{\pi,0}$ we cannot always approximate $B(f)$ by a monotonically decreasing computable sequence of rational numbers.

Theorem 10. *Let $f \in \mathcal{CB}^\infty_{\pi,0}$. Then we have $B(f) \notin \mathbb{R}_c$ if and only if there exist no monotonically decreasing computable sequence $\{\mu_n\}_{n\in\mathbb{N}}$ of rational numbers with $\lim_{n\to\infty} \mu_n = B(f)$.*

Remark 9. Theorem 10 answers the question when we have $B(f) \in \mathbb{R}_c$ for the signal of interest $f \in \mathcal{CB}^\infty_{\pi,0}$. Only in this case it is possible to find an algorithm that can compute $B(f)$ with arbitrary precision. Theorem 10 shows that the existence of a monotonically decreasing computable sequence that converges to $B(f)$ is sufficient and necessary for $B(f) \in \mathbb{R}_c$.

Proof of Theorem 10. "$\Rightarrow$": Let $B(f) \notin \mathbb{R}_c$. We use a proof by contradiction and assume that there exists a monotonically decreasing computable sequence $\{\mu_n\}_{n\in\mathbb{N}}$ of rational numbers with $\lim_{n\to\infty} \mu_n = B(f)$. Then the proof of Theorem 9 implies that the sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ from Theorem 8 converges effectively to $B(f)$, which in turn implies that $B(f) \in \mathbb{R}_c$. This is a contradiction.

"$\Leftarrow$": We do a proof by contradiction and assume that there exists no such monotonically decreasing computable sequence $\{\mu_n\}_{n\in\mathbb{N}}$ of rational numbers with $\lim_{n\to\infty} \mu_n = B(f)$, but we have $B(f) \in \mathbb{R}_c$. Since $B(f) \in \mathbb{R}_c$, we know from the first part of the proof of Theorem 9 that there exists a monotonically decreasing computable sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ of rational numbers that converges to $B(f)$. This is a contradiction. □

In Theorem 8, we have seen that $B(f)$ is the limit of a monotonically increasing computable sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ of rational numbers. However, for λ_n we only know that $\lambda_n < B(f)$, but not how far λ_n is from $B(f)$. Hence, this sequence does not help to answer the question of the minimum sampling rate. Moreover, from Theorem 10 we know that if $B(f) \notin \mathbb{R}_c$, then there exists no computable sequence of upper bounds for $B(f)$ that converges to $B(f)$. Thus, we do not only have $B(f) \notin \mathbb{R}_c$, but we also cannot algorithmically determine a sequence of upper bounds that converges to $B(f)$. Surprisingly this holds although the signal f is computable.

IX. Conclusion

Our analyses also address the following general question: Given computable, well-behaved "objects", which in our case are the computable bandlimited signals, do there exist physically relevant quantities associated with these objects that cannot always be computed? We have shown that the answer to this question is "yes". Computable signals $f \in \mathcal{CB}^2_\pi$ can be described arbitrarily well with respect to their time domain behavior and certain physical quantities, such as the energy $\|f\|^2_{\mathcal{B}^2_\pi}$ are computable, i.e., they can be algorithmically determined with arbitrary precision. However, there exist computable bandlimited signals $f \in \mathcal{CB}^2_\pi$ for which the actual bandwidth of the signal $B(f)$ is not computable. $B(f)$ determines the smallest symmetrical interval I in the frequency domain such that the entire energy of f is concentrated on I. Further, $B(f)$ characterizes the minimum sampling rate and therefore the sampling point sequences that uniquely describe the signal. It would be interesting to study which other signal properties are computable and which are not.

We can interpret the problem also from an optimization point of view. For signals $f \in \mathcal{CB}^1_\pi$, the actual bandwidth $B(f)$ is the smallest number $\sigma > 0$ such that

$$\int_{-\infty}^{\infty} |f(t)|^2 \, \mathrm{d}t = \frac{1}{2\pi} \int_{-\sigma}^{\sigma} |\hat{f}(\omega)|^2 \, \mathrm{d}\omega$$

Let

$$\Psi_f(\sigma) = \frac{1}{2\pi} \int_{-\sigma}^{\sigma} |\hat{f}(\omega)|^2 \, \mathrm{d}\omega$$

denote the functional that is relevant in this optimization problem. Since $\hat{f}$ is a computable continuous function, it follows that $\Psi_f \colon \mathbb{R} \to \mathbb{R}$ is a computable continuous function [26, Section 7, pp. 44]. We have

$$\|f\|^2_{\mathcal{B}^2_\pi} = \max_{\sigma\in\mathbb{R}} \Psi_f(\sigma) = \max_{\sigma\in\mathbb{R}} \frac{1}{2\pi} \int_{-\sigma}^{\sigma} |\hat{f}(\omega)|^2 \, \mathrm{d}\omega,$$

and, for $f \in \mathcal{CB}^1_\pi$, this maximum is computable. However, the smallest maximizer, i.e.,

$$B(f) = \min\left(\arg\max_{\sigma\in\mathbb{R}} \frac{1}{2\pi} \int_{-\sigma}^{\sigma} |\hat{f}(\omega)|^2 \, \mathrm{d}\omega \right)$$

is not computable in general. We formulated the optimization task for the signals in $\mathcal{CB}^1_\pi$ here. Since $\mathcal{CB}^1_\pi \subset \mathcal{CB}^2_\pi$, the problem equally exist for the larger set $\mathcal{CB}^2_\pi$.

One approach to prove non-computability for certain problems, is the technique presented in [26] by Pour-El. Please note that this technique works only for certain classes of linear operators $T \colon B_1 \to B_2$ that map between two Banach spaces B_1 and B_2. Further, it only gives the information that there exists an f_* in a computability structure of B_1 such that Tf_* is not computable in B_2. However, the non-computability of Tf_* in B_2 does not necessarily imply that $\|Tf_*\|_{B_2} \notin \mathbb{R}_c$. Nevertheless, for many practical applications, the norm is the essential quantity of interest, for example, the L^∞-norm measures the peak value and the L^2-norm the energy of a signal.

Since, in our case, the mapping $B \colon f \mapsto B(f)$ is non-linear, the Pour-El technique from [26] is not applicable. Further, the approach taken in the present paper also provides specific examples of signals $f_1 \in \mathcal{CB}^2_\pi$ and $f_3 \in \mathcal{CB}^1_\pi$ such that $B(f_1) \notin \mathbb{R}_c$ and $B(f_3) \notin \mathbb{R}_c$, respectively. To the best of our knowledge, the extension of the Pour-El theory to non-linear operators has not been studied so far, and the general answer to this question is open. With the results in the present paper, we provide first examples in this direction.

X. Further Open Problems

We have shown in Section VIII that, for $f \in \mathcal{CB}^\infty_{\pi,0}$, there exists a monotonically increasing computable sequence of rational numbers $\{\lambda_n\}_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} \lambda_n = B(f)$. Further, we have seen in Section VII that the set $\{f \in \mathcal{CB}^\infty_{\pi,0} \colon B(f) > \lambda\}$ is semi-decidable. In the proofs of these results, we used the specific structure of the signals in $\mathcal{B}^\infty_{\pi,0}$. It would be interesting to study both questions for a more general class of bandlimited signals, e.g., entire functions of exponential type.

An entire function f is completely determined by its Taylor coefficients $\{a_n\}_{n=0}^{\infty}$, according to

$$f(z) = \sum_{n=0}^{\infty} \frac{a_n}{n!} z^n, \tag{27}$$

where the series in (27) converges for all $z \in \mathbb{C}$, and for all $R > 0$, the convergence is uniform on $|z| \leq R$. This leads to the following definition of computability for entire functions of exponential type: An entire function of exponential type is called computable if the coefficients $\{a_n\}_{n=0}^{\infty}$ of the Taylor series form a computable sequence of computable numbers.

Note that all signals $f \in \mathcal{CB}_\pi^p$, $p \in [1, \infty] \cap \mathbb{R}_c$, are computable entire functions according to this definition. From the definition it is also clear that, for computable entire functions, we immediately have a computable representation on all compact subsets of the complex plane, and consequently on all compact subsets of the real axis. However, from the local behavior we cannot infer the behavior on the entire real axis.

For computable entire functions it is unclear whether the actual bandwidth is still the limit of a monotonically increasing computable sequence of computable rational numbers, and whether the set of computable entire functions of exponential type with $B(f) > \lambda$ is still semi-decidable. We conjecture that both statements do not hold in general for computable entire functions of exponential type. Another open problem is as follows: For entire functions of exponential type at most π it is unclear whether there exists a pair of Turing machines $\mathrm{TM}_{\underline{B}}$ and $\mathrm{TM}_{\overline{B}}$ such that, for every computable entire function of exponential type at most π with $B(f) \leq \pi$, we have

$$\mathrm{TM}_{\underline{B}}(f) \leq B(f) \leq \mathrm{TM}_{\overline{B}}(f).$$

Thus, $\mathrm{TM}_{\underline{B}}$ shall compute a lower bound on the actual bandwidth $B(f)$, and $\mathrm{TM}_{\overline{B}}$ an upper bound. We conjecture that the only Turing machines that have these properties, are the Turing machines that return the trivial bounds, i.e., 0 for $\mathrm{TM}_{\underline{B}}$, and π for $\mathrm{TM}_{\overline{B}}$. If these conjectures were true then we would have the interesting situation that the signal behavior on the real axis, e.g., a finite L^p-norm, determines the answer to certain questions of computability.

Regarding the general problem of computing the bandwidth, it would be interesting to find practically relevant Banach spaces of bandlimited functions with a computability structure, such that the actual bandwidth is always computable. Ideally, such spaces could be characterized by the time domain or frequency domain behavior of the signals.

A problem, similar to the computation of the bandwidth, is the problem of computing the period of a periodic computable continuous function. Please note that computing the period of certain computable continuous functions is the core task in Shor's famous algorithms [40] for factorizing natural numbers and computing the discrete logarithm, and also the only part of the algorithms that has to be implemented on a universal quantum computer. Interestingly, all candidates for a "post-quantum cryptography" that could already be broken, were broken by quantum algorithms that compute the period of certain computable functions. It seems as if finding periods of functions is the only class of well-investigated mathematical problems for which quantum algorithms could be developed that have a substantial complexity advantage over the best known classical algorithms. Hence, from this perspective, it would be meaningful to analyze the computability of the period of computable continuous functions.

There are many further interesting open problems and research directions related to computability in information theory, and we will discuss three of them next.

1) Finding extensions of the main technique of the Pour-El theory, such that the strongest possible result for non-computability, i.e., $\|Tf\|_{B_2} \notin \mathbb{R}_c$, can be shown, where T is some linear operator. Recently, numerous methods and operations have been analyzed with respect to computability on Turing machines, where the strongest form of non-computability has been considered. This strongest form of non-computability was shown, for example, for the interpolation of certain computable discrete-time signals with the Shannon sampling series in [41], for the downsampling of bandlimited signals in [42], and for the spectral factorization, the Wiener filter and prediction theory in [43]. All these examples have in common that for computable inputs non-computable outputs can be generated. Hence, the above operations are not always Turing computable, and tasks involving these operations cannot always be solved algorithmically. This is exactly the same non-computability behavior that we have shown in the present paper for the bandwidth, and correspondingly for the minimum sampling rate. Closely related to the phenomenon of non-computability is the fact that the existence of important objects in information theory often has to be proved by non-constructive methods. This situation seems to occur frequently in information theory, and it is surprising that, although even Shannon was forced to give a non-effective proof for the existence of a sequence of capacity achieving codes in his seminal work [44], the algorithmic construction of relevant objects has scarcely been studied so far. For the problem of constructing capacity achieving codes as function of the channel, it only recently could be shown that this task cannot by algorithmically solved by a Turing machine [45].

2) Studying optimization problems, in particular the computability of the optimizer, as discussed above. This aspect also plays a central role in information theory. A simple example is the computation of an optimal input distribution for a discrete memoryless channel that achieves the Shannon capacity. For this computation, different approaches have been suggested [46]–[50]. An analysis of the proofs in these publications, however, shows the approaches are non-constructive. That is, they do not allow an algorithmic control of the convergence speed. As a consequence, it is not clear whether the approaches from [46]–[50] are algorithms in the sense of Turing computability, i.e., in the sense of computability on digital computers. This needs to be considered, in particular since the computation of the optimal input distribution on digital computers has been the motivation for the publications [46], [47].

3) Analyzing computability in the context of time-variant channels [51]–[54]. Current results in this theory are usually

based on distribution theory and no guarantees for effective convergence are given. For additional discussions and results related to the capacity of channels, see for example [55].

Appendix A
Table of Symbols and Sets

Symbol	Meaning
$L^p(\Omega)$	Measurable, pth-power Lebesgue integrable functions on Ω
$\\|\cdot\\|_p$	L^p-norm: $\\|f\\|_p = (\int_\Omega \|f(t)\|^p \, \mathrm{d}t)^{1/p}$
$L^\infty(\Omega)$	Space of all measurable, essentially bounded functions on Ω
$\\|\cdot\\|_\infty$	L^∞ norm: $\\|f\\|_\infty = \operatorname{ess\,sup}_{t\in\Omega}\|f(t)\|$
$\mathcal{B}_\pi^p$	Bandlimited signals (bandwidth π) with finite L^p-norm
$\mathcal{B}_\pi^2$	Bandlimited signals (bandwidth π) with finite energy
$\mathcal{B}_\pi^\infty$	Bandlimited signals (bandwidth π) with finite L^∞-norm
$\mathcal{B}_{\pi,0}^\infty$	Bandlimited signals in $\mathcal{B}_\pi^\infty$ that vanish at infinity
$B(f)$	Actual bandwidth of a signal f
$\mathcal{CB}_\pi^p$	Set of all signals in $\mathcal{B}_\pi^p$ that are computable
$\mathcal{CB}_{\pi,0}^\infty$	Set of all signals in $\mathcal{B}_{\pi,0}^\infty$ that are computable
$\mathcal{A}$	A recursively enumerable nonrecursive set
$\mathcal{C}_{\mathrm{BW}}^1$	$\{f \in \mathcal{CB}_\pi^1 \colon B(f) \in \mathbb{R}_c\}$
$\mathcal{NC}_{\mathrm{BW}}^1$	$\mathcal{CB}_\pi^1 \setminus \mathcal{C}_{\mathrm{BW}}^1 = \{f \in \mathcal{CB}_\pi^1 \colon B(f) \notin \mathbb{R}_c\}$
$\mathcal{C}_{\mathrm{BW}}^2$	$\{f \in \mathcal{CB}_\pi^2 \colon B(f) \in \mathbb{R}_c\}$
$\mathcal{NC}_{\mathrm{BW}}^2$	$\mathcal{CB}_\pi^2 \setminus \mathcal{C}_{\mathrm{BW}}^2 = \{f \in \mathcal{CB}_\pi^2 \colon B(f) \notin \mathbb{R}_c\}$
$\mathcal{C}_>^1(\sigma)$	$\{f \in \mathcal{CB}_\pi^1 \colon B(f) > \sigma\}$
$\mathcal{C}_>^p(\sigma)$	$\{f \in \mathcal{CB}_\pi^p \colon B(f) > \sigma\}$
$\mathcal{C}_>^\infty(\sigma)$	$\{f \in \mathcal{CB}_{\pi,0}^\infty \colon B(f) > \sigma\}$
$\mathcal{C}_\leq^1(\sigma)$	$\{f \in \mathcal{CB}_\pi^1 \colon B(f) \leq \sigma\}$
$\mathcal{C}_<^1(\sigma)$	$\{f \in \mathcal{CB}_\pi^1 \colon B(f) < \sigma\}$

Appendix B
Minimum Bandwidth

For a given bandlimited signal f, we introduced in Section III the number $B(f) = \min\{\sigma \in \mathbb{R} \colon f \in \mathcal{B}_\sigma\}$ as the actual bandwidth of the signal f. Next, we will prove that this minimum indeed always exists.

Let $B(f) = \inf\{\sigma \in \mathbb{R} \colon f \in \mathcal{B}_\sigma\}$, i.e., let $B(f)$ be the infimum of all σ such that for all $\epsilon > 0$ there exists a constant $C(\epsilon)$ with

$$|f(z)| \leq C(\epsilon)\,\mathrm{e}^{(\sigma+\epsilon)|z|} \tag{28}$$

for all $z \in \mathbb{C}$. We will show that this infimum is actually attained, i.e., a minimum.

Let $\delta > 0$ be arbitrary but fixed. According to the properties of the infimum, there exists a σ_δ, such that for all $\epsilon > 0$ there exists a constant $C(\sigma_\delta, \epsilon)$ such that

$$|f(z)| \leq C(\sigma_\delta, \epsilon)\,\mathrm{e}^{(\sigma_\delta+\epsilon)|z|}$$

holds, and we have

$$\sigma_\delta \leq B(f) + \frac{\delta}{2}.$$

We choose $\epsilon = \delta/2$. Then it follows that

$$\begin{aligned} |f(z)| &\leq C(\sigma_\delta, \delta/2)\,\mathrm{e}^{(\sigma_\delta+\delta/2)|z|} \\ &\leq C(\sigma_\delta, \delta/2)\,\mathrm{e}^{(B(f)+\delta)|z|}. \end{aligned} \tag{29}$$

Hence, the number that is defined by the infimum, i.e., $B(f)$, satisfies the inequality (28), which shows that the infimum is actually attained.

Appendix C
Critical Nyquist Rate and Set of Uniqueness

In Section III we have seen that $\mathbb{Z}$ is a set of uniqueness for signals in $\mathcal{B}_\pi^p$, $1 \leq p < \infty$. Next, we will prove that this is in general no longer true if the sampling rate is reduced below the critical Nyquist rate. More specifically, we will show that if we have a signal $f \in \mathcal{B}_\pi^p$, $1 \leq p < \infty$, with $B(f) = \pi$, i.e., a signal $f \in \mathcal{B}_\pi^p$, $1 \leq p < \infty$, such that $f \notin \mathcal{B}_\sigma^p$ for all $\sigma < \pi$, then for all $\gamma < \pi$, f is not uniquely determined by its samples

$$\left\{ f\left(\frac{k\pi}{\gamma}\right) \right\}_{k\in\mathbb{Z}},$$

that is, for all $\gamma < \pi$, there exists a signal $g_\gamma \in \mathcal{B}_\pi^p$ such that

$$g_\gamma\left(\frac{k\pi}{\gamma}\right) = f\left(\frac{k\pi}{\gamma}\right), \quad k \in \mathbb{Z},$$

and $g_\gamma \neq f$.

We first prove the case $1 < p < \infty$. Let $f \in \mathcal{B}_\pi^p$, $1 < p < \infty$, be such that $f \notin \mathcal{B}_\sigma^p$ for all $\sigma < \pi$. We use an indirect proof and assume that the assertion is wrong, i.e., we assume that there exists a $\hat{\gamma} < \pi$ such that the samples

$$\left\{ f\left(\frac{k\pi}{\hat{\gamma}}\right) \right\}_{k\in\mathbb{Z}}$$

uniquely determine the signal f. Let $\hat{l}$ be a natural number, such that $\hat{l}\hat{\gamma} > \pi$. Then we have $f \in \mathcal{B}_{\hat{l}\hat{\gamma}}^p$, and it follows from the Plancherel–Pólya inequality that

$$\sum_{k=-\infty}^{\infty} \left| f\left(\frac{k\pi}{\hat{l}\hat{\gamma}}\right) \right|^p < \infty. \tag{30}$$

Since

$$\sum_{k=-\infty}^{\infty} \left| f\left(\frac{k\pi}{\hat{\gamma}}\right) \right|^p$$

is a partial sum of the sum in (30), we obtain

$$\sum_{k=-\infty}^{\infty} \left| f\left(\frac{k\pi}{\hat{\gamma}}\right) \right|^p < \sum_{k=-\infty}^{\infty} \left| f\left(\frac{k\pi}{\hat{l}\hat{\gamma}}\right) \right|^p < \infty.$$

Let

$$(\Gamma_N f)(t) = \sum_{k=-N}^{N} f\left(\frac{k\pi}{\hat{\gamma}}\right) \frac{\sin(\hat{\gamma}(t - \frac{k\pi}{\hat{\gamma}}))}{\hat{\gamma}(t - \frac{k\pi}{\hat{\gamma}})}.$$

It follows from the Plancherel–Pólya inequality that $\{\Gamma_N f\}_{N\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{B}_{\hat{\gamma}}^p$. Hence, there exists a signal $g \in \mathcal{B}_{\hat{\gamma}}^p$ with

$$g(t) = \lim_{N\to\infty} (\Gamma_N f)(t) = \sum_{k=-\infty}^{\infty} f\left(\frac{k\pi}{\hat{\gamma}}\right) \frac{\sin(\hat{\gamma}(t - \frac{k\pi}{\hat{\gamma}}))}{\hat{\gamma}(t - \frac{k\pi}{\hat{\gamma}})}, \tag{31}$$

where the convergence is in the $\mathcal{B}_{\hat{\gamma}}^p$-norm and consequently also globally uniformly on the real axis. Since $\hat{\gamma} < \pi$, g is also in $\mathcal{B}_\pi^p$. Further, we see from (31) that

$$g\left(\frac{k\pi}{\hat{\gamma}}\right) = f\left(\frac{k\pi}{\hat{\gamma}}\right), \quad k \in \mathbb{Z},$$

and it follows from our assumption that $g \equiv f$. Since $g \in \mathcal{B}^p_{\hat{\gamma}}$, this implies that $f \in \mathcal{B}^p_{\hat{\gamma}}$. However, this is a contradiction, because $f \notin \mathcal{B}^p_\sigma$ for all $\sigma < \pi$.

We now give the proof for $p = 1$. Let $f \in \mathcal{B}^1_\pi$ be such that $f \notin \mathcal{B}^1_\sigma$ for all $\sigma < \pi$. We use an indirect proof and assume that the assertion is wrong, i.e., we assume that there exists a $\hat{\gamma} < \pi$ such that the samples

$$\left\{ f\left(\frac{k\pi}{\hat{\gamma}}\right) \right\}_{k\in\mathbb{Z}}$$

uniquely determine the signal f. Let κ be such that $\hat{\gamma} < \kappa < \pi$, and let

$$g(t) = \sum_{k=-\infty}^{\infty} f\left(\frac{k\pi}{\hat{\gamma}}\right) K\left(t - \frac{k\pi}{\hat{\gamma}}\right), \tag{32}$$

where $K \in \mathcal{B}^1_\kappa$ is a kernel with cosine roll-of behavior in the frequency domain, i.e.,

$$\hat{K}(\omega) = \begin{cases} 1, & |\omega| \le (1-\alpha)\hat{\gamma} \\ \cos^2\left[\frac{\pi}{4\alpha\hat{\gamma}}(|\omega| - (1-\alpha)\hat{\gamma})\right], & (1-\alpha)\hat{\gamma} \le |\omega| \le (1+\alpha)\hat{\gamma} \\ 0, & \text{otherwise,} \end{cases}$$

where $0 < \alpha \le \min\{1, \kappa/\hat{\gamma} - 1\}$. Then $\hat{K}$ and is concentrated on the interval $[-\kappa, \kappa]$, and we have

$$K\left(\frac{k\pi}{\hat{\gamma}}\right) = \begin{cases} 1, & k = 0, \\ 0, & k \neq 0. \end{cases}$$

Let $\hat{l}$ be a natural number, such that $\hat{l}\hat{\gamma} > \pi$. Then we have $f \in \mathcal{B}^1_{\hat{l}\hat{\gamma}}$. Note that the left inequality of the Plancherel–Pólya inequality is also valid for $p = 1$ [18, Theorem 6.10, p. 50]. Thus, it follows from the Plancherel–Pólya inequality that

$$\sum_{k=-\infty}^{\infty} \left| f\left(\frac{k\pi}{\hat{l}\hat{\gamma}}\right) \right| < \infty. \tag{33}$$

Since

$$\sum_{k=-\infty}^{\infty} \left| f\left(\frac{k\pi}{\hat{\gamma}}\right) \right|$$

is a partial sum of the sum in (33), we obtain

$$\sum_{k=-\infty}^{\infty} \left| f\left(\frac{k\pi}{\hat{\gamma}}\right) \right| < \sum_{k=-\infty}^{\infty} \left| f\left(\frac{k\pi}{\hat{l}\hat{\gamma}}\right) \right| < \infty.$$

Hence, it follows that the sequence in (32) converges in $\mathcal{B}^1_\kappa$, and we have $g \in \mathcal{B}^1_\kappa \subset \mathcal{B}^1_\pi$. Further, we see from (32) that

$$g\left(\frac{k\pi}{\hat{\gamma}}\right) = f\left(\frac{k\pi}{\hat{\gamma}}\right), \quad k \in \mathbb{Z},$$

and it follows from our assumption that $g \equiv f$. Since $g \in \mathcal{B}^1_\kappa$, this implies that $f \in \mathcal{B}^1_\kappa$. However, this is a contradiction, because $f \notin \mathcal{B}^1_\sigma$ for all $\sigma < \pi$.

Appendix D
Auxiliary Result for the Proof of Theorem 1

Let a_l, $l \in \mathbb{N}$, be the same numbers as defined in the proof of Theorem 1. We show that

$$\frac{\sin(a_l(t-l))}{\pi(t-l)} \tag{34}$$

is a computable function in $\mathcal{B}^2_\pi$ for all $l \in \mathbb{N}$.

Let $l \in \mathbb{N}$ be arbitrary but fixed. For $N \in \mathbb{N}$,

$$\sum_{k=-N}^{N} \frac{\sin(a_l(k-l))}{\pi(k-l)} \frac{\sin(\pi(t-k))}{\pi(t-k)}$$

is an elementary computable function in $\mathcal{B}^2_\pi$, because the coefficients

$$\left\{ \frac{\sin(a_l(k-l))}{\pi(k-l)} \right\}_{k=-N}^{N}$$

are computable numbers, which follows from the fact that sinc is a computable continuous function [26]. Using Parseval's relation [18, p. 24], we see that

$$\int_{-\infty}^{\infty} \left| \frac{\sin(a_l(t-l))}{\pi(t-l)} - \sum_{k=-N}^{N} \frac{\sin(a_l(k-l))}{\pi(k-l)} \frac{\sin(\pi(t-k))}{\pi(t-k)} \right|^2 \mathrm{d}t$$
$$= \sum_{|k|>N} \left| \frac{\sin(a_l(k-l))}{\pi(k-l)} \right|^2.$$

For $N > l$, we have

$$\begin{aligned} \sum_{|k|>N} \left| \frac{\sin(a_l(k-l))}{\pi(k-l)} \right|^2 &\le \frac{2}{\pi^2} \sum_{k=N+1}^{\infty} \frac{1}{(k-l)^2} \\ &= \frac{2}{\pi^2} \sum_{k=N-l+1}^{\infty} \frac{1}{k^2} \\ &< \frac{2}{\pi^2} \int_{N-l}^{\infty} \frac{1}{\tau^2} \, \mathrm{d}\tau \\ &= \frac{2}{\pi^2(N-l)}. \end{aligned}$$

This shows that

$$\left\{ \sum_{k=-N}^{N} \frac{\sin(a_l(k-l))}{\pi(k-l)} \frac{\sin(\pi(t-k))}{\pi(t-k)} \right\}_{N\in\mathbb{N}}$$

converges effectively to (34) in the $\mathcal{B}^2_\pi$-norm, which implies that (34) is a computable function in $\mathcal{B}^2_\pi$.

Appendix E
Computability of g_δ in the Proof of Theorem 2

In this section we show that, for $\delta \in (0, \pi) \cap \mathbb{R}_c$,

$$\begin{aligned} g_\delta(t) &= \frac{1}{2\pi} \int_{-\delta}^{\delta} \left(1 - \frac{|\omega|}{\delta}\right) \mathrm{e}^{i\omega t} \, \mathrm{d}\omega \\ &= \frac{\delta}{2\pi} \left(\frac{\sin(\frac{\delta t}{2})}{\frac{\delta t}{2}} \right)^2, \quad t \in \mathbb{R}, \end{aligned}$$

is a computable function in $\mathcal{B}_\pi^1$. We have

$$\begin{aligned} t g_\delta(t) &= \frac{2}{\delta} \sin\left(\frac{\delta t}{2}\right) \frac{\sin(\frac{\delta t}{2})}{\pi t} \\ &= \frac{1}{i\delta}\left(\mathrm{e}^{i\frac{\delta t}{2}} - \mathrm{e}^{-i\frac{\delta t}{2}}\right)\frac{\sin(\frac{\delta t}{2})}{\pi t}. \end{aligned}$$

Taking the Fourier transform, we obtain

$$i\hat{g}_\delta'(\omega) = \frac{1}{i\delta}\left[\mathbf{1}_{[-\frac{\delta}{2},\frac{\delta}{2}]}\left(\omega - \frac{\delta}{2}\right) - \mathbf{1}_{[-\frac{\delta}{2},\frac{\delta}{2}]}\left(\omega + \frac{\delta}{2}\right)\right],$$

where $\mathbf{1}_A$ denotes the indicator function of the set A, and it follows that

$$\hat{g}_\delta'(\omega) = \frac{1}{\delta}\left[\mathbf{1}_{[-\frac{\delta}{2},\frac{\delta}{2}]}\left(\omega + \frac{\delta}{2}\right) - \mathbf{1}_{[-\frac{\delta}{2},\frac{\delta}{2}]}\left(\omega - \frac{\delta}{2}\right)\right].$$

We see that $\hat{g}_\delta' \in L^\infty[-\pi,\pi]$ and $\hat{g}_\delta'(\omega) = 0$ for all $|\omega| > \delta$. Let

$$c_k(\hat{g}_\delta') = \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{g}_\delta'(\omega)\,\mathrm{e}^{-i\omega k}\,\mathrm{d}\omega, \quad k \in \mathbb{Z},$$

and

$$(S_N\hat{g}_\delta')(\omega) = \sum_{k=-N}^{N} c_k(\hat{g}_\delta')\,\mathrm{e}^{i\omega k}, \quad N \in \mathbb{N}.$$

We define

$$\hat{p}_N(\omega) = \begin{cases} \sum_{k=-N}^{N} g_\delta(k)\left(\mathrm{e}^{-i\omega k} - \mathrm{e}^{-i\pi k}\right), & |\omega| \le \pi \\ 0, & |\omega| > \pi. \end{cases}$$

Then we have

$$\hat{p}_N(\omega) = (S_N\hat{g}_\delta)(\omega) - \sum_{k=-N}^{N} g_\delta(k)\,\mathrm{e}^{-i\pi k}$$

for $|\omega| \le \pi$, which corresponds to

$$p_N(t) = \sum_{k=-N}^{N} g_\delta(k)\frac{\sin(\pi(t-k))}{\pi(t-k)} - \sum_{k=-N}^{N} g_\delta(k)\,\mathrm{e}^{-i\pi k}\,\frac{\sin(\pi t)}{\pi t}$$

in the time domain. Thus, we see that $\{p_N\}_{N\in\mathbb{N}}$ is a computable sequence of elementary computable functions in $\mathcal{B}_\pi^1$. We further have

$$\hat{p}_N'(\omega) = (S_N\hat{g}_\delta)'(\omega) = (S_N\hat{g}_\delta')(\omega),$$

and consequently,

$$\begin{aligned} \frac{t}{i}(g_\delta(t) - p_N(t)) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} (\hat{g}_\delta'(\omega) - \hat{p}_N'(\omega))\,\mathrm{e}^{i\omega t}\,\mathrm{d}\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} (\hat{g}_\delta'(\omega) - (S_N\hat{g}_\delta')(\omega))\,\mathrm{e}^{i\omega t}\,\mathrm{d}\omega. \end{aligned}$$

Applying Plancherel's identity, we obtain

$$\int_{-\infty}^{\infty} t^2|g_\delta(t) - p_N(t)|^2\,\mathrm{d}t = \frac{1}{2\pi}\int_{-\pi}^{\pi} |\hat{g}_\delta'(\omega) - (S_N\hat{g}_\delta')(\omega)|^2\,\mathrm{d}\omega. \tag{35}$$

We call a function $f \in L^2[-\pi,\pi]$ computable if there exists a computable sequence of trigonometric polynomials with computable coefficients, that converges effectively to f in the L^2-norm. To show that $\hat{g}_\delta'$ is a computable function in $L^2[-\pi,\pi]$ it suffices to show that the indicator function is a computable function in $L^2[-\pi,\pi]$. This is indeed the case, since the Fourier coefficients of the indicator function decay as $1/k$. Since, $\hat{g}_\delta'$ is a computable function in $L^2[-\pi,\pi]$, it follows that $S_N\hat{g}_\delta'$ converges to $\hat{g}_\delta'$ effectively in $L^2[-\pi,\pi]$. See Lemma 3 in Appendix G for details. From (35), we see that

$$\lim_{N\to\infty}\int_{-\infty}^{\infty} t^2|g_\delta(t) - p_N(t)|^2\,\mathrm{d}t = 0, \tag{36}$$

where the convergence is effective in N. We have

$$\begin{aligned} \int_1^\infty |g_\delta(t) - p_N(t)|\,\mathrm{d}t &= \int_1^\infty \frac{1}{t}t|g_\delta(t) - p_N(t)|\,\mathrm{d}t \\ &\le \left(\int_1^\infty \frac{1}{t^2}\,\mathrm{d}t\right)^{\frac{1}{2}}\left(\int_1^\infty t^2|g_\delta(t) - p_N(t)|^2\,\mathrm{d}t\right)^{\frac{1}{2}} \\ &= \left(\int_1^\infty t^2|g_\delta(t) - p_N(t)|^2\,\mathrm{d}t\right)^{\frac{1}{2}}, \end{aligned} \tag{37}$$

and, similarly,

$$\int_{-\infty}^{-1} |g_\delta(t) - p_N(t)|\,\mathrm{d}t \le \left(\int_{-\infty}^{-1} t^2|g_\delta(t) - p_N(t)|^2\,\mathrm{d}t\right)^{\frac{1}{2}}. \tag{38}$$

Further, we have

$$\int_{-1}^{1} |g_\delta(t) - p_N(t)|\,\mathrm{d}t \le 2\max_{|t|\le 1}|g_\delta(t) - p_N(t)|. \tag{39}$$

Since

$$\begin{aligned} |g_\delta(t) - p_N(t)| &\le \frac{1}{2\pi}\int_{-\pi}^{\pi} |\hat{g}_\delta(\omega) - \hat{p}_N(\omega)|\,\mathrm{d}\omega \\ &\le \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |\hat{g}_\delta(\omega) - \hat{p}_N(\omega)|^2\,\mathrm{d}\omega\right)^{\frac{1}{2}} \\ &= \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \left|\hat{g}_\delta(\omega) - (S_N\hat{g}_\delta)(\omega) + \sum_{k=-N}^{N} g_\delta(k)\,\mathrm{e}^{-i\pi k}\right|^2\mathrm{d}\omega\right)^{\frac{1}{2}} \\ &\le \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |\hat{g}_\delta(\omega) - (S_N\hat{g}_\delta)(\omega)|^2\,\mathrm{d}\omega\right)^{\frac{1}{2}} \\ &\quad + \left|\sum_{k=-N}^{N} g_\delta(k)\,\mathrm{e}^{-i\pi k}\right|, \end{aligned}$$

where the first term converges effectively to zero according to Lemma 3 in Appendix G, and the second term converges effectively to zero, because $\hat{g}_\delta(\pi) = 0$ and

$$\begin{aligned} \left|\hat{g}_\delta(\pi) - \sum_{k=-N}^{N} g_\delta(k)\,\mathrm{e}^{-i\pi k}\right| &\le \sum_{|k|>N} |g_\delta(k)| \\ &\le \frac{2}{\delta\pi}\sum_{|k|>N}\frac{1}{k^2} \\ &\le \frac{4}{\delta\pi}\frac{1}{N}, \end{aligned}$$

we see that

$$\lim_{N\to\infty}\max_{|t|\le 1}|g_\delta(t) - p_N(t)| = 0, \tag{40}$$

where the convergence is effective in N. Combining (37)–(39) and using (36) as well as (40), we obtain

$$\lim_{N\to\infty}\int_{-\infty}^{\infty} |g_\delta(t) - p_N(t)|\,\mathrm{d}t = 0,$$

where the convergence is effective in N. Hence, $\{p_N\}_{N\in\mathbb{N}}$ is a computable sequence of elementary computable functions in $\mathcal{B}_\pi^1$ that converges effectively to g_δ in the L^1-norm. This shows that $g_\delta \in \mathcal{CB}_\pi^1$.

Appendix F
Computability of g in the Proof of Theorem 5

In this section we show that the function g in Theorem 5, which was defined as

$$g(t) = \frac{\left(f(\frac{t}{2}) - f(0)\right)^2}{t^2}, \quad t \in \mathbb{R},$$

is computable in $\mathcal{B}_\pi^1$, i.e., that $g \in \mathcal{CB}_\pi^1$ for all $f \in \mathcal{CB}_\pi^p$, $p \in [1,\infty) \cap \mathbb{R}_c$, and all $f \in \mathcal{CB}_{\pi,0}^\infty$. We do the proof for $f \in \mathcal{CB}_\pi^p$, $p \in [1,\infty) \cap \mathbb{R}_c$, here. The proof for $f \in \mathcal{CB}_{\pi,0}^\infty$ is done in an analogue way, when all occurrences of the $\mathcal{B}_\pi^p$-norm below are replaced by the $\mathcal{B}_\pi^\infty$-norm. Since $f \in \mathcal{CB}_\pi^p$, there exists a sequence $\{f_n\}_{n\in\mathbb{N}}$ of elementary computable functions

$$f_n(t) = \sum_{k=-M_n}^{M_n} a_k(n) \frac{\sin(\pi(t-k))}{\pi(t-k)},$$

where M_n and $a_k(n)$ are computable functions with respect to n, and a recursive function $\xi\colon \mathbb{N} \to \mathbb{N}$ such that for all $N \in \mathbb{N}$ we have

$$\|f - f_n\|_{\mathcal{B}_\pi^p} \le \frac{1}{2^N}, \tag{41}$$

for all $n \ge \xi(N)$. For $n \in \mathbb{N}$, let

$$g_n(t) = \frac{\left(f_n(\frac{t}{2}) - f_n(0)\right)^2}{t^2}, \quad t \in \mathbb{R}.$$

Next, we show that $g_n \in \mathcal{CB}_\pi^1$. We have

$$\begin{aligned} &f_n(\tfrac{t}{2}) - f_n(0) \\ &= \sum_{\substack{k=-M_n\\k\neq 0}}^{M_n} a_k(n) \frac{\sin(\pi(\frac{t}{2}-k))}{\pi(\frac{t}{2}-k)} - a_0(n)\left(1 - \frac{\sin(\frac{\pi}{2}t)}{\frac{\pi}{2}t}\right), \end{aligned}$$

and, consequently,

$$\begin{aligned} g_n(t) &= \frac{(f_n(\frac{t}{2}) - f_n(0))^2}{t^2} \\ &= \frac{1}{t^2}\Bigg(\sum_{\substack{k_1=-M_n\\k_1\neq 0}}^{M_n} \sum_{\substack{k_2=-M_n\\k_2\neq 0}}^{M_n} a_{k_1}(n) a_{k_2}(n) \times \\ &\qquad \times \frac{\sin(\pi(\frac{t}{2}-k_1))}{\pi(\frac{t}{2}-k_1)} \frac{\sin(\pi(\frac{t}{2}-k_2))}{\pi(\frac{t}{2}-k_2)} \\ &\quad - 2a_0(n)\left(1 - \frac{\sin(\frac{\pi}{2}t)}{\frac{\pi}{2}t}\right) \sum_{\substack{k=-M_n\\k\neq 0}}^{M_n} a_k(n) \frac{\sin(\pi(\frac{t}{2}-k))}{\pi(\frac{t}{2}-k)} \\ &\quad + (a_0(n))^2 \left(1 - \frac{\sin(\frac{\pi}{2}t)}{\frac{\pi}{2}t}\right)^2 \Bigg). \end{aligned} \tag{42}$$

We now show that the following three functions are in $\mathcal{CB}_\pi^1$:

$$\frac{1}{t^2} \frac{\sin(\pi(\frac{t}{2}-k_1))}{\pi(\frac{t}{2}-k_1)} \frac{\sin(\pi(\frac{t}{2}-k_2))}{\pi(\frac{t}{2}-k_2)} \tag{43}$$

for $k_1, k_2 \in \mathbb{Z} \setminus \{0\}$,

$$\frac{1}{t^2}\left(1 - \frac{\sin(\frac{\pi}{2}t)}{\frac{\pi}{2}t}\right) \frac{\sin(\pi(\frac{t}{2}-k))}{\pi(\frac{t}{2}-k)} \tag{44}$$

for $k \in \mathbb{Z} \setminus \{0\}$, and

$$\frac{1}{t^2}\left(1 - \frac{\sin(\frac{\pi}{2}t)}{\frac{\pi}{2}t}\right)^2. \tag{45}$$

Since $\mathcal{CB}_\pi^1$ is a linear subspace, it then follows that g_n, which is a finite linear combination of the functions (43)–(45), is in $\mathcal{CB}_\pi^1$. We start with the analysis of (43) for $k_1, k_2 \in \mathbb{Z} \setminus \{0\}$. For convenience, we introduce the abbreviation

$$q(t,k_2,k_2) = \frac{1}{t^2} \frac{\sin(\pi(\frac{t}{2}-k_1))}{\pi(\frac{t}{2}-k_1)} \frac{\sin(\pi(\frac{t}{2}-k_2))}{\pi(\frac{t}{2}-k_2)}.$$

We have

$$\begin{aligned} \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{q}''(\omega,k_1,k_2)\, \mathrm{e}^{i\omega t} \, \mathrm{d}\omega &= -t^2 q(t,k_1,k_2) \\ &= -\frac{\sin(\pi(\frac{t}{2}-k_1))}{\pi(\frac{t}{2}-k_1)} \frac{\sin(\pi(\frac{t}{2}-k_2))}{\pi(\frac{t}{2}-k_2)}. \end{aligned}$$

Since

$$\frac{\sin(\frac{\pi}{2}(t-2k_1))}{\frac{\pi}{2}(t-2k_1)} = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} 2\,\mathrm{e}^{-i\omega 2k_1}\,\mathrm{e}^{i\omega t}\, \mathrm{d}\omega,$$

we see that

$$\hat{q}''(\omega,k_1,k_2) = -\frac{1}{2\pi}(u_{k_1} * u_{k_2})(\omega)$$

where

$$\hat{u}_l(\omega) = \mathbf{1}_{[-\frac{\pi}{2},\frac{\pi}{2}]}(\omega) 2\,\mathrm{e}^{-i2l\omega}, \quad l = k_1, k_2.$$

$\mathbf{1}_A$ denotes the indicator function of the set A. We have

$$\begin{aligned} &\hat{q}''(\omega,k_1,k_2) \\ &= -\frac{2}{\pi}\int_{-\infty}^{\infty} \mathbf{1}_{[-\frac{\pi}{2},\frac{\pi}{2}]}(\omega_1)\,\mathrm{e}^{-i2k_1\omega_1} \times \\ &\qquad\qquad \times \mathbf{1}_{[-\frac{\pi}{2},\frac{\pi}{2}]}(\omega-\omega_1)\,\mathrm{e}^{-i2k_2(\omega-\omega_1)}\, \mathrm{d}\omega_1 \\ &= -\frac{2}{\pi}\int_{-\pi/2}^{\pi/2} \mathrm{e}^{-i2k_1\omega_1}\,\mathbf{1}_{[-\frac{\pi}{2},\frac{\pi}{2}]}(\omega-\omega_1)\,\mathrm{e}^{-i2k_2(\omega-\omega_1)}\, \mathrm{d}\omega_1 \\ &= -\frac{2}{\pi}\int_{\omega-\pi/2}^{\omega+\pi/2} \mathrm{e}^{-i2k_1(\omega-\xi)}\,\mathbf{1}_{[-\frac{\pi}{2},\frac{\pi}{2}]}(\xi)\,\mathrm{e}^{-i2k_2\xi}\, \mathrm{d}\xi \\ &= \begin{cases} \frac{2}{\pi}\,\mathrm{e}^{-i(k_1+k_2)\omega}\,\frac{\sin[(|\omega|-\pi)(k_2-k_1)]}{k_2-k_1}, & |\omega| \le \pi, \\ 0, & |\omega| > \pi. \end{cases} \end{aligned}$$

Hence, $\hat{q}''(\,\cdot\,,k_1,k_2)$ and consequently $\hat{q}'(\,\cdot\,,k_1,k_2)$, are computable continuous functions. For $N \in \mathbb{N}$, we consider

$$\hat{\rho}_N(\omega) = \begin{cases} \sum_{k=-N}^{N} q(\omega,k_1,k_2)(\mathrm{e}^{-i\omega k} - \mathrm{e}^{-i\pi k}), & |\omega| \le \pi, \\ 0, & |\omega| > \pi. \end{cases}$$

Let

$$c_r(\hat{q}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{q}(\omega,k_1k_2)\,\mathrm{e}^{-i\omega r}\, \mathrm{d}\omega, \quad r \in \mathbb{Z},$$

and

$$(S_N\hat{q})(\omega) = \sum_{r=-N}^{N} c_r(\hat{q})\,\mathrm{e}^{i\omega r}, \quad N \in \mathbb{N}.$$

For $|\omega| < \pi$, we have

$$\hat{\rho}_N'(\omega) = (S_N \hat{q})'(\omega) = (S_N \hat{q}')(\omega). \tag{46}$$

Further, we have $\hat{\rho}_N'(\omega) = 0$ for $|\omega| > \pi$. Hence, we see that $\hat{\rho}_N' \in L^\infty(\mathbb{R})$. It follows, using Plancherel's theorem, that

$$\begin{aligned} \frac{1}{2\pi}\int_{-\pi}^{\pi} |\hat{\rho}_N'(\omega)|^2 \,\mathrm{d}\omega &= \int_{-\infty}^{\infty} |it\rho_N(t)|^2 \,\mathrm{d}t \\ &= \int_{-\infty}^{\infty} t^2 |\rho_N(t)|^2 \,\mathrm{d}t. \end{aligned}$$

Due to the linearity of the Fourier transform, we obtain

$$\begin{aligned} &\frac{1}{2\pi}\int_{-\pi}^{\pi} |\hat{q}'(t,k_1,k_2) - (S_N\hat{q}')(\omega)|^2 \,\mathrm{d}\omega \\ &\quad = \frac{1}{2\pi}\int_{-\pi}^{\pi} |\hat{q}'(t,k_1,k_2) - \hat{\rho}_N'(\omega)|^2 \,\mathrm{d}\omega \\ &\quad = \int_{-\infty}^{\infty} t^2 |q(t,k_1,k_2) - \rho_N(t)|^2 \,\mathrm{d}t, \end{aligned} \tag{47}$$

where we used (46) in the first equality. Since, $\hat{q}'(\,\cdot\,,k_1,k_2)$ is a computable continuous function it follows from Lemma 2 in Appendix G that $\hat{q}'(\,\cdot\,,k_1,k_2)$ is a computable function in $L^2[-\pi,\pi]$, and as a consequence of Lemma 3 in Appendix G, that $S_N\hat{q}'$ converges to $\hat{q}'$ effectively in $L^2[-\pi,\pi]$. Thus, from (47), we see that

$$\lim_{N\to\infty} \int_{-\infty}^{\infty} t^2 |q(t,k_1,k_2) - \rho_N(t)|^2 \,\mathrm{d}t = 0, \tag{48}$$

where the convergence is effective in N. We have

$$\begin{aligned} &\int_{1}^{\infty} |q(t,k_1,k_2) - \rho_N(t)| \,\mathrm{d}t \\ &\quad = \int_{1}^{\infty} \frac{1}{t} t|q(t,k_1,k_2) - \rho_N(t)| \,\mathrm{d}t \\ &\quad \le \left(\int_{1}^{\infty} \frac{1}{t^2}\,\mathrm{d}t\right)^{\frac{1}{2}} \left(\int_{1}^{\infty} t^2|q(t,k_1,k_2) - \rho_N(t)|^2 \,\mathrm{d}t\right)^{\frac{1}{2}} \\ &\quad = \left(\int_{1}^{\infty} t^2|q(t,k_1,k_2) - \rho_N(t)|^2 \,\mathrm{d}t\right)^{\frac{1}{2}}, \end{aligned} \tag{49}$$

and, similarly,

$$\begin{aligned} &\int_{-\infty}^{-1} |q(t,k_1,k_2) - \rho_N(t)| \,\mathrm{d}t \\ &\qquad \le \left(\int_{-\infty}^{-1} t^2|q(t,k_1,k_2) - \rho_N(t)|^2 \,\mathrm{d}t\right)^{\frac{1}{2}}, \end{aligned} \tag{50}$$

Further, we have

$$\int_{-1}^{1} |q(t,k_1,k_2) - \rho_N(t)| \,\mathrm{d}t \le 2 \max_{|t|\le 1} |q(t,k_1,k_2) - \rho_N(t)| \tag{51}$$

and, by the same calculation as in Appendix E,

$$\lim_{N\to\infty} \max_{|t|\le 1} |q(t,k_1,k_2) - \rho_N(t)| = 0, \tag{52}$$

where the convergence is effective in N. Combining (49)–(51) and using (48) as well as (52), we obtain

$$\lim_{N\to\infty} \int_{-\infty}^{\infty} |q(t,k_1,k_2) - \rho_N(t)| \,\mathrm{d}t = 0,$$

where the convergence is effective in N. Since

$$\begin{aligned} \rho_N(t) = &\sum_{r=-N}^{N} q(r,k_1,k_2) \frac{\sin(\pi(t-r))}{\pi(t-r)} \\ &- \sum_{r=-N}^{N} q(r,k_1,k_2)\, \mathrm{e}^{-i\pi r} \frac{\sin(\pi t)}{\pi t}, \end{aligned}$$

we see that $\{\rho_N\}_{N\in\mathbb{Z}}$ is a computable sequence of elementary computable functions in $\mathcal{B}_\pi^1$ that converges effectively to q in the L^1-norm. Hence, we have $q \in \mathcal{CB}_\pi^1$. The functions (44) and (45) are treated similarly. The exact calculations are omitted due to space constraints. It follows that $g_n \in \mathcal{CB}_\pi^1$ for all $n \in \mathbb{N}$. Next, we show the effective convergence of g_n to g in the L^1-norm. To this end, we treat both integrals on the right-hand side of

$$\begin{aligned} &\int_{-\infty}^{\infty} |g(t) - g_n(t)| \,\mathrm{d}t \\ &\quad = \int_{|t|\le 1} |g(t) - g_n(t)| \,\mathrm{d}t + \int_{|t|>1} |g(t) - g_n(t)| \,\mathrm{d}t \end{aligned} \tag{53}$$

separately.

We start with the second integral. We have

$$\begin{aligned} &\int_{1}^{\infty} |g(t) - g_n(t)| \,\mathrm{d}t \\ &\quad = \int_{1}^{\infty} \left| \frac{(f(\frac{t}{2}) - f(0))^2}{t^2} - \frac{(f_n(\frac{t}{2}) - f_n(0))^2}{t^2} \right| \mathrm{d}t \\ &\quad \le \max_{t\in\mathbb{R}} |(f(\tfrac{t}{2}) - f(0))^2 - (f_n(\tfrac{t}{2}) - f_n(0))^2| \int_{1}^{\infty} \frac{1}{t^2}\,\mathrm{d}t \\ &\quad = \max_{t\in\mathbb{R}} |(f(\tfrac{t}{2}) - f(0))^2 - (f_n(\tfrac{t}{2}) - f_n(0))^2| \end{aligned} \tag{54}$$

and

$$\begin{aligned} &|(f(\tfrac{t}{2}) - f(0))^2 - (f_n(\tfrac{t}{2}) - f_n(0))^2| \\ &\quad = |(f(\tfrac{t}{2}))^2 - (f_n(\tfrac{t}{2}))^2 - (f_n(0))^2 + (f(0))^2 \\ &\qquad\qquad - 2f(\tfrac{t}{2})f(0) + 2f_n(\tfrac{t}{2})f_n(0)| \\ &\quad \le |(f(\tfrac{t}{2}))^2 - (f_n(\tfrac{t}{2}))^2| + |(f(0))^2 - (f_n(0))^2| \\ &\qquad\qquad + 2|f(\tfrac{t}{2})f(0) - f_n(\tfrac{t}{2})f_n(0)|. \end{aligned} \tag{55}$$

For the first summand in (55), we obtain

$$\begin{aligned} &|(f(\tfrac{t}{2}))^2 - (f_n(\tfrac{t}{2}))^2| \\ &\quad = |(f(\tfrac{t}{2}) - f_n(\tfrac{t}{2}))(f(\tfrac{t}{2}) + f_n(\tfrac{t}{2}))| \\ &\quad \le |f(\tfrac{t}{2}) - f_n(\tfrac{t}{2})| \cdot |f(\tfrac{t}{2}) + f_n(\tfrac{t}{2})| \\ &\quad \le |f(\tfrac{t}{2}) - f_n(\tfrac{t}{2})| \left(\|f\|_\infty + \|f_n\|_\infty\right) \\ &\quad \le \|f - f_n\|_\infty \left(\|f\|_\infty + \|f_n\|_\infty\right). \end{aligned}$$

Since $\|f\|_\infty$ and $\|f_n\|_\infty$ are computable numbers, there exists a computable number N_0 such that

$$\big|\|f\|_\infty - \|f_n\|_\infty\big| < \frac{1}{2}$$

for all $n \ge N_0$. It follows that

$$\|f_n\|_\infty < \|f\|_\infty + \frac{1}{2}$$

for all $n \ge N_0$. Hence, we see that

$$|(f(\tfrac{t}{2}))^2 - (f_n(\tfrac{t}{2}))^2| \le C_1 \|f - f_n\|_\infty \tag{56}$$

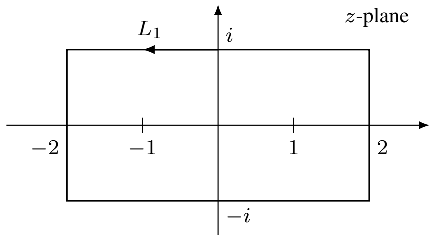

Fig. 3. Contour L_1 in the complex plane.

for all $n \geq N_0$, where

$$C_1 = 2\|f\|_\infty + \frac{1}{2}$$

is a computable constant. Using the same calculation, we obtain for the second summand in (55) that

$$|(f(0))^2 - (f_n(0))^2| \leq C_1 \|f - f_n\|_\infty \tag{57}$$

for all $n \geq N_0$, For the third summand, we have

$$\begin{aligned} &|f(\tfrac{t}{2})f(0) - f_n(\tfrac{t}{2})f_n(0)| \\ &\quad = |f(\tfrac{t}{2})f(0) - f_n(\tfrac{t}{2})f(0) + f_n(\tfrac{t}{2})f(0) - f_n(\tfrac{t}{2})f_n(0)| \\ &\quad \leq |f(0)| \cdot |f(\tfrac{t}{2}) - f_n(\tfrac{t}{2})| + |f_n(\tfrac{t}{2})| \cdot |f(0) - f_n(0)| \\ &\quad \leq \|f\|_\infty \cdot \|f - f_n\|_\infty + \|f_n\|_\infty \cdot \|f - f_n\|_\infty \\ &\quad \leq C_1 \|f - f_n\|_\infty \end{aligned} \tag{58}$$

for all $n \geq N_0$. Combining (54)–(58), it follows that

$$\begin{aligned} \int_{|t|\geq 1} |g(t) - g_n(t)| \,\mathrm{d}t &= 2\int_1^\infty |g(t) - g_n(t)| \,\mathrm{d}t \\ &\leq 8C_1 \|f - f_n\|_\infty \\ &\leq 8C_1 \|f - f_n\|_{\mathcal{B}_\pi^p}, \end{aligned}$$

and, using (41), that

$$\int_{|t|\geq 1} |g(t) - g_n(t)| \,\mathrm{d}t \leq \frac{8C_1}{2^N}$$

for all $n \geq \max\{\xi(N), N_0\}$.

Next, we treat the first integral in (53). For all $h \in \mathcal{B}_{\pi,0}^\infty$ we have, as a consequence of the Phragmén–Lindelöf principle, the inequality

$$|h(\tfrac{z}{2})| \leq \|h\|_\infty \,\mathrm{e}^{\frac{\pi}{2} y_0}$$

for all $z = x + iy$ with $|y| \leq y_0$ [35, Remark 2, p. 38], [56, Theorem 11, p. 82]. Hence, it follows that

$$|f(\tfrac{z}{2}) - f_n(\tfrac{z}{2})| \leq \|f - f_n\|_\infty \,\mathrm{e}^{\frac{\pi}{2} y_0}$$

for all $z = x + iy$ with $|y| \leq y_0$. Let L_1 be the contour that is depicted in Fig. 3, and let $t \in [-1, 1]$. Then, according to Cauchy's integral formula [18, p. 91], we have

$$\begin{aligned} |g(t) - g_n(t)| &= \left| \frac{1}{2\pi} \oint_{L_1} \frac{g(z) - g_n(z)}{z - t} \,\mathrm{d}z \right| \\ &\leq \max_{z\in L_1} |g(z) - g_n(z)| \frac{1}{2\pi} \oint_{L_1} \frac{1}{|z - t|} \,|\mathrm{d}z|. \end{aligned}$$

Since $|z - t| \geq 1$ for all $z \in L_1$, it follows that

$$\oint_{L_1} \frac{1}{|z - t|} \,|\mathrm{d}z| \leq 12.$$

Thus, we obtain

$$|g(t) - g_n(t)| \leq \frac{6}{\pi} \max_{z\in L_1} |g(z) - g_n(z)|.$$

For $z \in L_1$ we have, using a similar calculation as before, that

$$\begin{aligned} |g(z) - g_n(z)| &= \frac{1}{|z|^2} \left| (f(\tfrac{z}{2}) - f(0))^2 - (f_n(\tfrac{z}{2}) - f_n(0))^2 \right| \\ &\leq \left| (f(\tfrac{z}{2}) - f(0))^2 - (f_n(\tfrac{z}{2}) - f_n(0))^2 \right| \\ &\leq \mathrm{e}^{\pi} \|f - f_n\|_\infty^2 + \|f - f_n\|_\infty^2 \\ &\qquad + 2C_1 \,\mathrm{e}^{\frac{\pi}{2}} \|f - f_n\|_\infty \\ &\leq (1 + \mathrm{e}^{\pi}) \|f - f_n\|_{\mathcal{B}_\pi^p}^2 + 2C_1 \,\mathrm{e}^{\frac{\pi}{2}} \|f - f_n\|_{\mathcal{B}_\pi^p} \end{aligned}$$

for all $n \geq N_0$. Hence, we have

$$\begin{aligned} &\int_{-1}^1 |g(t) - g_n(t)| \,\mathrm{d}t \leq 2 \max_{t\in[-1,1]} |g(t) - g_n(t)| \\ &\quad \leq \frac{12}{\pi} \left[(1 + \mathrm{e}^{\pi}) \|f - f_n\|_{\mathcal{B}_\pi^p}^2 + 2C_1 \,\mathrm{e}^{\frac{\pi}{2}} \|f - f_n\|_{\mathcal{B}_\pi^p} \right] \end{aligned}$$

for all $n \geq N_0$. It follows that

$$\begin{aligned} \int_{-1}^1 |g(t) - g_n(t)| \,\mathrm{d}t &\leq \frac{12}{\pi} \left[(1 + \mathrm{e}^{\pi}) \frac{1}{2^{2N}} + 2C_1 \,\mathrm{e}^{\frac{\pi}{2}} \frac{1}{2^N} \right] \\ &\leq C_2 \frac{1}{2^N} \end{aligned}$$

for all $n \geq \max\{\xi(N), N_0\}$, where

$$C_2 = \frac{12}{\pi} \left[(1 + \mathrm{e}^{\pi}) + 2C_1 \,\mathrm{e}^{\frac{\pi}{2}} \right]$$

is a computable constant. Finally, we see that for all $N \in \mathbb{N}$ we have

$$\int_{-\infty}^\infty |g(t) - g_n(t)| \,\mathrm{d}t \leq C_3 \frac{1}{2^N}$$

for all $n \geq \max\{\xi(N), N_0\}$, where $C_3 = 8C_1 + C_2$ is a computable constant. Thus, it follows that $\{g_n\}_{n\in\mathbb{N}}$ converges effectively to g in the L^1-norm, and consequently that g is computable in $\mathcal{B}_\pi^1$.

Appendix G
Computability of Fourier Series

In this section we give a result about the L^2-convergence of the Fourier series for computable continuous 2π-periodic functions.

The following lemma is obvious, nevertheless its proof is included for completeness.

Lemma 2. *Let f be a computable continuous 2π-periodic function. Then f is a computable function in $L^2[-\pi, \pi]$.*

Proof. Let f be an arbitrary computable continuous 2π-periodic function. Since f is a computable continuous 2π-periodic function, there exists a computable sequence of computable trigonometric polynomials $\{p_N\}_{N\in\mathbb{N}}$ that effectively approximates f [26], i.e., we have

$$\lim_{N\to\infty} \max_{\omega\in[-\pi,\pi]} |f(\omega) - p_N(\omega)| = 0,$$

where the convergence is effective. Since

$$\|f - p_N\|_{L^2[-\pi,\pi]} = \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(\omega) - p_N(\omega)|^2 \, \mathrm{d}\omega \right)^{\frac{1}{2}} \leq \max_{\omega \in [-\pi,\pi]} |f(\omega) - p_N(\omega)|,$$

it follows that f is computable in $L^2[-\pi, \pi]$. □

For $N \in \mathbb{N}$, let

$$(S_N f)(\omega) = \sum_{k=-N}^{N} c_k(f) \, \mathrm{e}^{i\omega k},$$

where

$$c_k(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\omega) \, \mathrm{e}^{-i\omega k} \, \mathrm{d}\omega.$$

Lemma 3. *Let f be a computable function in $L^2[-\pi, \pi]$. Then*

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(\omega) - (S_N f)(\omega)|^2 \, \mathrm{d}\omega$$

converges effectively to zero as N tends to infinity.

Proof. Let f be an arbitrary computable function in $L^2[-\pi, \pi]$. Then we have

$$\|f\|^2_{L^2[-\pi,\pi]} = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(\omega)|^2 \, \mathrm{d}\omega \in \mathbb{R}_c.$$

Since

$$\sum_{k=-N}^{N} |c_k(f)|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |(S_N f)(\omega)|^2 \, \mathrm{d}\omega,$$

it follows that

$$\lim_{N \to \infty} \sum_{k=-N}^{N} |c_k(f)|^2 = \|f\|^2_{L^2[-\pi,\pi]} \in \mathbb{R}_c.$$

Hence

$$\left\{ \sum_{k=-N}^{N} |c_k(f)|^2 \right\}_{N \in \mathbb{N}}$$

is a monotonically increasing computable sequence of computable numbers that converges to a computable number. Thus the convergence has to be effective [26, p. 20, Corollary 2a]. As a consequence

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(\omega) - (S_N f)(\omega)|^2 \, \mathrm{d}\omega = \sum_{|k|>N} |c_k(f)|^2$$

converges effectively to zero as N tends to infinity. □

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